Modal team logics for modelling Free Choice inference

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Helsinki Logic Seminar

Overview

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Motivation: Free Choice (fc)

Aloni: Bilateral state-based modal logic (*BSML*) accounts for fc

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Natural deduction axiomatizations

Free choice (fc) inference

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A possible formalization:

(†) $\quad \square(\ \square\ ) \to \square$

Free choice (fc) inference

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A possible formalization:

(†) □ (□) → □

Problem:

1:	□ *p*	
2:	□ (*p* □ *q*)	(1, classical modal logic)
3:	□ *q*	(2, †)

Bilateral State-based Modal Logic

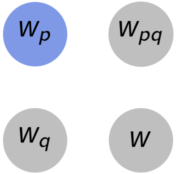
Team semantics for modal logic

$$M = (W, R, V)$$

standard Kripke semantics

$$M, w \models$$

$$w \in W$$

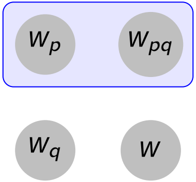


$$w_p \models p$$

state-based/team semantics

$$M, s \models$$

$$s \subseteq W$$



$$\{w_p, w_{pq}\} \models p$$

Bilateralism

“ is assertable in s” s (

“ is rejectable in s” s)

Bilateralism

" is assertable in s"	s (
" is rejectable in s"	s)

Bilateral negation

s (¬	□□	s)
s) ¬	□□	s (

Syntax of *BSML*:

$$[illegible] = p \; [illegible] \; \neg \; [illegible] \; (\; [illegible] \;) \; [illegible] \; (\; [illegible] \;) \; [illegible] \; [illegible] \; [illegible] \; \mathrm{ne}$$

Syntax of *BSML*:

$$□□= p □ ¬ \quad □ (\quad □ \quad) □ (\quad □ \quad) □ \; □ \quad □ \mathrm{ne}$$

Semantics (( )

$s \;(\; p$	□□	$□w □ s □ w □ V(p)$
$s \;(\; ¬$	□□	$s \;\;)$
$s \;(\quad □$	□□	$s \;(\quad$ and $s \;($
$s \;(\quad □$	□□	$□t; t' □ t □ t' = s$ and $t \;(\quad$ and $t' \;($
$s \;(\; □$	□□	$□w □ s □ □t □ R[w] □ t \neq □$ and $t \;($
$s \;(\; \mathrm{ne}$	□□	$s \neq □$

$R[w] = \{v □ W □ wRv\}$

Syntax of *BSML*:

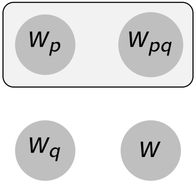
□□= *p* □ ¬ □ (□) □ (□) □ □ □ ne

Semantics (( )

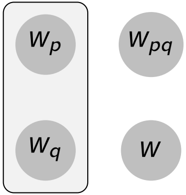
s (*p*	□□	□*w* □ *s* □ *w* □ *V*(*p*)
s (¬	□□	*s*)
s (□	□□	*s* (and *s* (
s (□	□□	□*t*; *t*′ □ *t* □ *t*′ = *s* and *t* (and *t*′ (
s (□	□□	□*w* □ *s* □ □*t* □ *R*[*w*] □ *t* ≠ □ and *t* (
s (ne	□□	*s* ≠ □

R[*w*] = {*v* □ *W* □ *wRv*}

$$s \models p \iff \forall w \in s : w \in V(p)$$



(a) $s \models p$



(b) $s \not\models p$

Syntax of *BSML*:

$$= p \ \neg \ (\ \square \ ) \ (\ \square \ ) \ \square \ \text{ne}$$

Semantics (( )

s (p	□□	□w s w $V(p)$
s (¬	□□	s)
s (□	□□	s (and s (
s (□	□□	□$t; t'$ t □ t' = s and t (and t' (
s (□	□□	□w s □t $R[w]$ $t \neq$ □ and t (
s (ne	□□	$s \neq$ □

$R[w] = \{v \ W \ wRv\}$

Syntax of *BSML*:

$$□□= p □ ¬ \quad □ (\quad □ \quad) □ (\quad □ \quad) □ \; □ \quad □ \, \mathrm{ne}$$

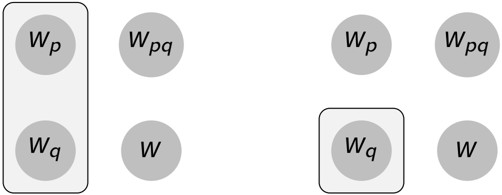
Semantics (( )

s (p	□□	□w □ s □ w □ $V(p)$
s (¬	□□	s)
s (□	□□	s (and s (
s (□	□□	□$t; t'$ □ t □ t' = s and t (and t' (
s (□	□□	□w □ s □ □t □ $R[w]$ □ $t \neq$ □ and t (
s (ne	□□	$s \neq$ □

$R[w] = \{v$ □ W □ $wRv\}$

Tensor disjunction □

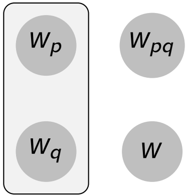
s (□ □□ □$t; t'$ □ t □ $t' = s$ and
t (and
t' (



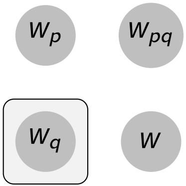
(a) s (p □ q (b) s (p □ q

The non-emptiness atom ne

$$s \models \text{ne} \iff s \neq \emptyset$$



(a) $s \models (p \wedge \text{ne}) \vee (q \wedge \text{ne})$



(b) $s \not\models (p \wedge \text{ne}) \vee (q \wedge \text{ne})$

The modality ◇

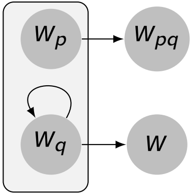
$$R[w] = \{v \in W \mid wRv\}$$

$$s \models \Diamond\varphi \iff \forall w \in s \; \exists t \subseteq R[w] : t \neq \emptyset \text{ and } t \models \varphi$$

The modality □

$R[w] = \{v$ □ W □ $wRv\}$

s (□ □□ □w □ s □ □t □ $R[w]$ □ $t \neq$ □ and t (



s (□q since

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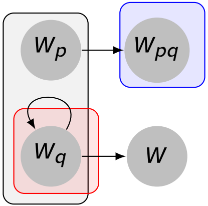


s (□q since

$\{w_q\}$ □ $R[w_q]$
$\{w_q\}$ (q

and

$\{w_{pq}\}$ □ $R[w_p]$
$\{w_{pq}\}$ (q

Syntax of *BSML*:

$$= p \mid \neg \quad (\quad) \quad (\quad) \quad \text{ne}$$

Semantics (( )

s (p		w s w $V(p)$
s ($\neg$		s)
s (		s (and s (
s (		$t; t'$ t $t' = s$ and t (and t' (
s (		w s t $R[w]$ $t \neq$ and t (
s (ne		$s \neq$

$R[w] = \{v \quad W \quad wRv\}$

Accounting for fc

The empty team □ supports contradictions such as p □ $\neg p$

ne represents lack of contradiction

Accounting for fc

The empty team $\emptyset$ supports contradictions such as $p \wedge \neg p$
ne represents lack of contradiction

fc is caused by an intrusion of the pragmatic principle "avoid stating a contradiction" (ne) into meaning composition:

Accounting for fc

The empty team $\emptyset$ supports contradictions such as $p \wedge \neg p$
NE represents lack of contradiction

fc is caused by an intrusion of the pragmatic principle "avoid stating a contradiction" (NE) into meaning composition:

$$
\begin{aligned}
p^+ &:= p \wedge \text{NE} \\
(\neg \alpha)^+ &:= \neg \alpha^+ \wedge \text{NE} \\
(\alpha \vee \beta)^+ &:= (\alpha^+ \vee \beta^+) \wedge \text{NE} \\
(\alpha \wedge \beta)^+ &:= (\alpha^+ \wedge \beta^+) \wedge \text{NE} \\
(\Diamond \alpha)^+ &:= \Diamond \alpha^+ \wedge \text{NE}
\end{aligned}
$$

You may have coffee or tea.

$\rightsquigarrow$ You may have coffee and you may have tea.

$$(\Diamond(c \vee t))^+ \rightsquigarrow \Diamond c \wedge \Diamond t$$

You may have coffee or tea.

□You may have coffee and you may have tea.

$$(\square(c \,\square\, t))^{+} \;(\; \square c \,\square\, \square t$$

i.e. $\square(((c \,\square\, \mathrm{ne}) \,\square\, (t \,\square\, \mathrm{ne})) \,\square\, \mathrm{ne}) \,\square\, \mathrm{ne} \;(\; \square c \,\square\, \square t$

You may have coffee or tea.

⇝ You may have coffee and you may have tea.

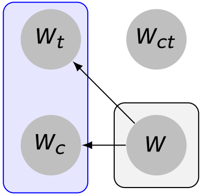
$$(\Diamond(c \vee t))^{+} \models \Diamond c \wedge \Diamond t$$

$$\text{i.e. } \Diamond(((c \wedge \text{ne}) \vee (t \wedge \text{ne})) \wedge \text{ne}) \wedge \text{ne} \models \Diamond c \wedge \Diamond t$$

$$\text{i.e. } \Diamond((c \wedge \text{ne}) \vee (t \wedge \text{ne})) \models \Diamond c \wedge \Diamond t$$

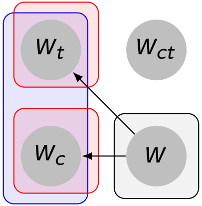
$\square\ ((c\ \square\ \mathrm{ne})\ \square\ (t\ \square\ \mathrm{ne}))$ ($\square\ c\ \square\ \square t$

□ ((*c* □ ne) □ (*t* □ ne)) (□ *c* □ □*t*



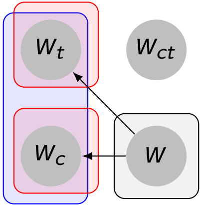
$\{w\}$ (□((*c* □ ne) □ (*t* □ ne)) since

□ $((c$ □ $\text{ne})$ □ $(t$ □ $\text{ne}))$ (□ c □ □t



$\{w\}$ (□$((c$ □ $\text{ne})$ □ $(t$ □ $\text{ne}))$ since $\{w_c\}$ (c and $\{w_t\}$ (t

□ ((*c* □ ne) □ (*t* □ ne)) (□ *c* □ □*t*



$\{w\}$ (□((*c* □ ne) □ (*t* □ ne)) since $\{w_c\}$ (*c* and $\{w_t\}$ (*t*

for the same reason, $\{w\}$ (□*c* □ □*t*

BSML$^{\square}$: *BSML* with the global disjunction □

$$s \; (\quad \square \qquad \square\square \quad s \; (\quad \text{or } s \; ($$

BSML$^{□}$: *BSML* with the global disjunction □

$$s \;(\quad □ \qquad □□ \quad s \;(\quad \text{or } s \;($$

BSML$^{□}$: *BSML* with the emptiness/circle operator □

$$s \;(□ \qquad □□ \quad s \;(\quad \text{or } s = □$$

BSML□: *BSML* with the global disjunction □

$$s \; (\quad \square \qquad \square\square \quad s \; (\quad \text{or } s \; ($$

BSML□: *BSML* with the emptiness/circle operator □

$$s \; (\; \square \qquad \square\square \quad s \; (\quad \text{or } s = \square$$

For classical formulas (no ne; □; □):

$$s \; (\qquad \square\square \; \square w \, \square \, s \, \square \, \{w\} \; (\qquad \square\square \; \square w \, \square \, s \, \square \, w \; ($$

Semantics ( ))

s) p	□□	□w □ s □ w □ $V(p)$
s) ¬	□□	s (
s) □	□□	□$t; t'$ □ t □ t' = s and t) and t')
s) □	□□	s) and s)
s) □	□□	s) and s)
s) □	□□	□w □ s □ $R[w]$)
s) ne	□□	s = □
s) □	□□	s)

Semantics ( ))

s) p	□□	□w □ s □ w □ $V(p)$
s) ¬	□□	s (
s) □	□□	□$t; t'$ □ t □ t' = s and t) and t')
s) □	□□	s) and s)
s) □	□□	s) and s)
s) □	□□	□w □ s □ $R[w]$)
s) ne	□□	s = □
s) □	□□	s)

□ □= ¬ □ ¬

s (□ □□ □w □ s □ $R[w]$ (

Semantics ( ))

s) p □□ □w □ s □ w □ $V(p)$

s) ¬ □□ s (

s) □ □□ □$t;t'$ □ t □ t' = s and t) and t')

s) □ □□ s) and s)

s) □ □□ s) and s)

s) □ □□ □w □ s □ $R[w]$)

s) ne □□ s = □

s) □ □□ s)

¬ behaves classically when is classical

□ □= ¬ □ ¬

s (□ □□ □w □ s □ $R[w]$ (

¬¬ ¬(□) ¬ □¬

¬ne p □ ¬p ¬(□) ¬ □¬

¬ □ ¬ ¬(□) ¬ □¬

¬ □ □¬

You may not have coffee or tea.

$\rightsquigarrow$ You may not have coffee and you may not have tea.

$$\neg\Diamond(c \vee t) \rightarrow (\neg\Diamond c \wedge \neg\Diamond t)$$

You may not have coffee or tea.

$\square$ You may not have coffee and you may not have tea.

$$\neg \square (c \square t) \to (\neg \square c \square \neg \square t)$$

In *BSML*: $(\neg \square (b \square c))^{+}$ ($\neg \square b \square \neg \square c$.

For classical : □(□ ne)

Using □ we can define a function which cancels pragmatic enrichment:

p^{-}	□=	□ p
ne^{-}	□=	□ ne
$(\neg\ )^{-}$	□=	□ $\neg\ ^{-}$
(□ $)^{-}$	□=	□ $^{-}$ □ □ $^{-}$
(□ $)^{-}$	□=	□ $^{-}$ □ □ $^{-}$
(□ $)^{-}$	□=	□ □ $^{-}$

For classical □ ($^{+})^{-}$

Closure properties

φ is *downward closed*: $[M,s \vDash \varphi$ and $t \subseteq s] \implies M,t \vDash \varphi$

φ is *union closed*: $[M,s \vDash \varphi$ for all $s \in S \neq \emptyset] \implies M,\bigcup S \vDash \varphi$

φ has the *empty team property*: $M,\emptyset \vDash \varphi$ for all M

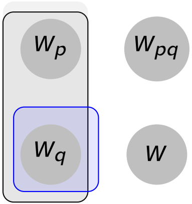
φ is *flat*: $M,s \vDash \varphi \iff M,\{w\} \vDash \varphi$ for all $w \in s$

flat $\iff$ downward closed & union closed & empty team property

Classical formulas are flat

Classical formulas are flat

Formulas with ne may lack downward closure and the empty team property:

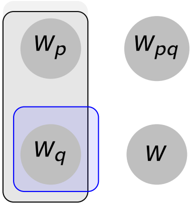


$\{w_p, w_q\} \models (p \wedge \text{ne}) \vee (q \wedge \text{ne})$

$\{w_q\} \not\models (p \wedge \text{ne}) \vee (q \wedge \text{ne})$

Classical formulas are flat

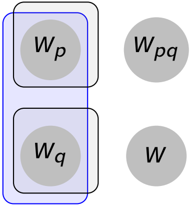
Formulas with ne may lack downward closure and the empty team property:



$\{w_p; w_q\}$ ($(p$ □ ne$)$ □ $(q$ □ ne$)$

$\{w_q\}$ * $(p$ □ ne$)$ □ $(q$ □ ne$)$

Formulas with □ may lack union closure:



$\{w_p\}$ (p □ q

$\{w_q\}$ (p □ q

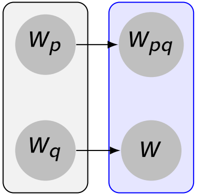
$\{w_p; w_q\}$ * p □ q

The modal dependence logic modalities $\Box$ and $\Diamond$

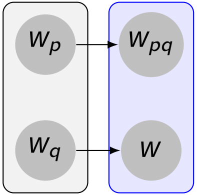
t is a successor team of s

$sRt \iff t \subseteq R[s]$ and $R[w] \cap t \neq \emptyset$ for all $w \in s$

$R[s] = \{v \in W \mid \exists w \in s : wRv\}$



The modal dependence logic modalities □ and □



t is a successor team of s

sRt □ □□ *t* □ *R*[*s*] and *R*[*w*] ∩ *t* ≠ □ for all *w* □ *s*

R[*s*] = {*v* □ *W* □ □*w* □ *s* □ *wRv*}

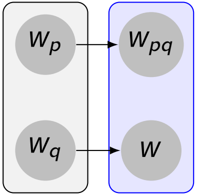
s (□ □□ □*t* □ *sRt* and *t* (

s (□ □□ *R*[*s*] (

s (□ □□ □*w* □ *s* □ □*t* □ *R*[*w*] □ *t* (and *t* ≠ □

s (□ □□ □*w* □ *s* □ *R*[*w*] (

The modal dependence logic modalities □ and □



t is a successor team of s

sRt □ □□ t □ $R[s]$ and $R[w] \cap t \neq$ □ for all w □ s

$R[s] = \{v$ □ W □ □w □ s □ $wRv\}$

s (□ □□ □t □ sRt and t (

s (□ □□ $R[s]$ (

s (□ □□ □w □ s □ □t □ $R[w]$ □ t (and $t \neq$ □

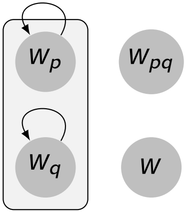
s (□ □□ □w □ s □ $R[w]$ (

If is downward closed, □ (□ and □ (□

If is union closed and has the empty team property, □ (□ and □ (□

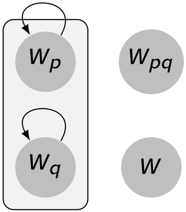
If is flat, □ □ and □ □

Aloni's free choice explanation does not work with $\Box$:



$$sRs \text{ and } s \models (p \wedge \text{ne}) \vee (q \wedge \text{ne})$$
$$\text{Therefore } s \models \Box((p \wedge \text{ne}) \vee (q \wedge \text{ne}))$$

Aloni's free choice explanation does not work with $\Box$:



sRs and $s \models (p \wedge \text{ne}) \vee (q \wedge \text{ne})$
Therefore $s \models \Box((p \wedge \text{ne}) \vee (q \wedge \text{ne}))$

s is the the only successor team of s and $s \not\models p$
Therefore $s \not\models \Box p$ so $\Box((p \wedge \text{ne}) \vee (q \wedge \text{ne})) \not\models \Box p \wedge \Box q$

Expressive Power

Fix a finite set of proposition symbols Φ

Pointed team model: $(M; s)$ where M is a model over Φ; s is a team on M

Team property: set of pointed team models

[illegible] [illegible]$= \{(M; s)$ [illegible] $M; s$ ([illegible] $\}$

Expressive Power

Fix a finite set of proposition symbols Φ

Pointed team model: $(M; s)$ where M is a model over Φ; s is a team on M

Team property: set of pointed team models

$\|\varphi\| = \{(M; s) \mid M; s \models \varphi\}$

Theorem

$$\{\|\varphi\| \mid \varphi \in BSML^{\Box}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

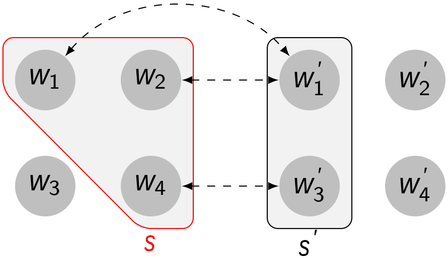
Team bisimulation:

$s \;-_k\; s'$ ▯ □□

Team bisimulation:

$s \rightleftharpoons_k s' \iff$

forth: $\forall w \in s \; \exists w' \in s' \; w \rightleftharpoons_k w'$



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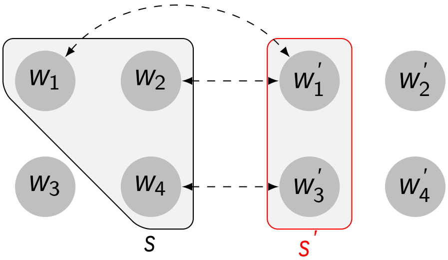

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$s \rightleftharpoons_k s' \;:\Longleftrightarrow$

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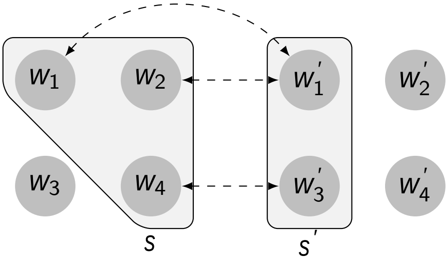
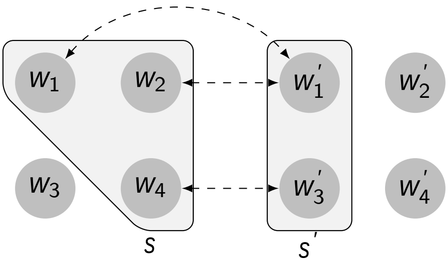
$s \rightleftharpoons s' \;:\Longleftrightarrow\; \forall k \in \mathbb{N} : s \rightleftharpoons_k s'$



Team bisimulation:

$s \sim_k s' \iff$
forth: $\forall w \in s\ \exists w' \in s' : w \sim_k w'$
back: $\forall w' \in s'\ \exists w \in s : w \sim_k w'$

$s \sim s' \iff \forall k \in \mathbb{N} : s \sim_k s'$



Theorem (bisimulation invariance)

$$s \sim_k s' \implies s \equiv^k s'$$
$$s \sim s' \implies s \equiv s'$$

Property P is *invariant under team k-bisimulation*:

$$[(M, s) \in P \text{ and } M, s \rightleftharpoons_k M', s'] \implies (M', s') \in P$$

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Characteristic formulas for worlds
(Hintikka formulas):

$$\chi^0_{M;w} := \bigwedge\{p \mid w \in V(p)\} \wedge \bigwedge\{\neg p \mid w \notin V(p)\} \quad (p \in \Phi)$$

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$$\chi^{k+1}_{M;w} := \chi^{k}_{M;w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^{k}_{M;v} \wedge \Box \bigvee_{v \in R[w]} \chi^{k}_{M;v}$$

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$$w' \models \chi^{k}_{w} \iff w \rightleftharpoons_{k} w'$$

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Characteristic formulas for teams:

$$\Theta^k_{M,s} := \bot \quad \text{if } s = \emptyset \quad (\bot := p \wedge \neg p)$$

$$\Theta^k_{M,s} := \bigvee_{w \in s} (\chi^k_{M,w} \wedge \text{NE}) \quad \text{if } s \neq \emptyset$$

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$$s' \models \Theta^{k}_{s} \iff s \rightleftarrows_{k} s'$$

$$\Theta_s^k = \square_{w \square s} \left({}_w^k \square \text{ne} \right) \qquad s' \left(\Theta_s^k \quad \square\square \quad s -_k s' \right.$$

$\Theta_s^k = \text{[illegible]}_{w \text{[illegible]} s} ({}^{k}_{w} \text{[illegible] ne})$ $\qquad$ $s' (\Theta_s^k$ [illegible] $s -_k s'$

Proof.

Case 1 : $s =$ [illegible].

Then $\Theta_s^k =$ [illegible] and

$s -_k s'$ [illegible] $s' =$ [illegible] [illegible] $s' ($ [illegible].

$$\Theta_s^k = \underset{w \square s}{\square} (\ {}^k_w \square \text{ne}) \qquad s' \ (\ \Theta_s^k \ \square\square \ s \ \text{-}_k \ s'$$

Proof.

Case 1 : $s = \square$.
Then $\Theta_s^k = \square$ and
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Case 2: $s \neq \square$.

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Therefore $s' (\underset{w \square s}{\square} (\;{}^k_w \square \text{ne})$, i.e. $s' (\Theta_s^k$.

$$\Theta^k_s = \underset{w \square s}{\square} (\ ^k_w \square \text{ne}) \qquad s' (\Theta^k_s \ \square\square \ s -_k s'$$

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$s' (\Theta^k_s$ implies there are $s'_w \square s'$ ($\square w \square s$) s.t.
$s'_w (\ ^k_w \square$ ne and $s' = \square_{w \square s} s'_w$.

$$\Theta_s^k = \underset{w \square s}{\square} (\ {}^k_w \square \text{ne}) \qquad\qquad s' \ (\ \Theta_s^k \ \square\square \ s \ -_k \ s'$$

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Let $w \square s$. Then there is some $s'_w \square s'$ such that $s'_w \ (\ {}^{k}_{w} \square$ ne.

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$\square w' \square s' \square \square w \square s \square w \text{-}_k w'$
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Let $w' \square s'$. Then for some $w \square s$ we have $w' \square s'_w$. Since $s'_w (\;{}^k_w \square \text{ne}$, we have $s'_w (\;{}^k_w$.
So $w' (\;{}^k_w$ whence $w \text{-} w'$.

Let $w \square s$. Then there is some $s'_w \square s'$ such that $s'_w (\;{}^k_w \square \text{ne}$. Since $s'_w (\text{ ne}$, there is some $w' \square s'_w$. Then $w' (\;{}^k_w$ and so $w \text{-}_k w'$.

Characteristic formulas for properties
(disjunctive normal form):

for P invariant under k-bisimulation:

$$M', s' \models \bigvee_{(M,s)\in P} \theta_s^k \iff (M', s') \in P$$

Characteristic formulas for properties
(disjunctive normal form):

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Theorem

$$\{\|\varphi\| \mid \varphi \in BSML^{\Box}\}$$
$$=$$
$$\{\text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

Property P is *union closed*:

$$\{(M; s_i) \mid i \in I \neq \emptyset\} \subseteq P \implies (M; \bigcup_{i \in I} s_i) \in P$$

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Theorem

$$\{\|\phi\| \mid \phi \in BSML^{\oslash}\}$$
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$$\mathbb{U} := \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$$

BSML is union closed, but not expressively complete for
$\mathbb{U} = \{P \mid P$ is union closed and invariant under k-bisimulation for some $k \in \mathbb{N}\}$.

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Lemma

For ∈ *BSML*: has the empty team property ⟺ is downward closed.

$BSML$ is union closed, but not expressively complete for $\mathbb{U} = \{P \mid P$ is union closed and invariant under k-bisimulation for some $k \in \mathbb{N}\}$.

Lemma

For $\varphi \in BSML$: φ has the empty team property $\iff$ φ is downward closed.

Consider $\|(p \wedge \text{ne}) \vee (\neg p \wedge \text{ne})\| \cup \|\bot\| \in \mathbb{U}$.

Assume $\|\varphi\| = \|(p \wedge \text{ne}) \vee (\neg p \wedge \text{ne})\|$ for $\varphi \in BSML$.

BSML is union closed, but not expressively complete for
$U = \{P \mid P$ is union closed and invariant under k-bisimulation for some $k \in \mathbb{N}\}$.

Lemma

For $\varphi \in BSML$: φ has the empty team property $\iff$ φ is downward closed.

Consider $\|(p \wedge \mathrm{ne}) \vee (\neg p \wedge \mathrm{ne})\|$ [illegible] $\in U$.
Assume $\|\varphi\| = \|(p \wedge \mathrm{ne}) \vee (\neg p \wedge \mathrm{ne})\|$ for $\varphi \in BSML$.
If $\{w_p, w_{\neg p}\} \models (p \wedge \mathrm{ne}) \vee (\neg p \wedge \mathrm{ne})$, then $\{w_p, w_{\neg p}\} \models \varphi$.

BSML is union closed, but not expressively complete for
$\mathbb{U} = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$.

Lemma

For $\varphi \in BSML$: φ has the empty team property $\iff$ φ is downward closed.

Consider $\|(p \wedge \text{ne}) \vee (\neg p \wedge \text{ne})\| \cup \{\emptyset\} \in \mathbb{U}$.
Assume $\|\varphi\| = \|(p \wedge \text{ne}) \vee (\neg p \wedge \text{ne})\|$ for $\varphi \in BSML$.
If $\{w_p, w_{\neg p}\} \models (p \wedge \text{ne}) \vee (\neg p \wedge \text{ne})$, then $\{w_p, w_{\neg p}\} \models \varphi$.
By downward closure $\{w_p\} \models \varphi$.
$\{w_p\} \notin \|(p \wedge \text{ne}) \vee (\neg p \wedge \text{ne})\| \cup \{\emptyset\}$, a contradiction.

BSML is union closed, but not expressively complete for
U = {*P* □ *P* is union closed and invariant under *k*-bisimulation for some *k* □ N}.

Lemma

For □ *BSML*: has the empty team property □□ is downward closed.

Consider □□(*p* □ ne) □ (¬*p* □ ne)□□ □ □□□□ □ U.
Assume □□ □□ = □□(*p* □ ne) □ (¬*p* □ ne)□□ for □ *BSML*.
If $\{w_p; w_{\neg p}\}$ ((*p* □ ne) □ (¬*p* □ ne), then $\{w_p; w_{\neg p}\}$ (.
By downward closure $\{w_p\}$ (.
$\{w_p\}$ □ □□(*p* □ ne) □ (¬*p* □ ne)□□ □ □□□□, a contradiction.

In *BSML*□: □□(*p* □ ne) □ (¬*p* □ ne)□□ □ □□□□ = □□ □ ((*p* □ ne) □ (¬*p* □ ne))□□

$$s' \ (\ \Theta_s^k \qquad \square\square \qquad s \ - _k \ s'$$

$$s' \ (\ \square\Theta_s^k \qquad \square\square \qquad s \ - _k \ s' \text{ or } s = \square$$

$$s' \;(\; \Theta_s^k \qquad \square\square \qquad s \;\text{-}_k\; s'$$

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Characteristic formulas for union-closed properties with the empty team property:

$$M'; s' \;(\; \underset{(M;s)\square P}{\square} \square\Theta_s^k \;\; \square\square \;\; (M'; s') \,\square\, P$$

$$s' \; (\; \Theta_s^k \qquad \square\square \qquad s \;\text{-}_k\; s'$$

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Characteristic formulas for union-closed properties without the empty team property:

$$M'; s' \; (\; (\underset{(M;s)\square P}{\square} \square\Theta_s^k) \square \text{ne} \;\; \square\square \;\; (M';s') \square P$$

$$s' \;(\; \Theta_s^k \quad \square\square \quad s \;-_k\; s'$$

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Theorem

$$\{\square\square \;\; \square\square\,\square \quad \square\, BSML^{\square}\}$$
$$=$$
$$\{P \,\square\, P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \,\square\, \mathbb{N}\}$$

For P union-closed; with the empty team property; invariant under k-bisimulation:

$$M', s' \models \bigvee_{(M,s)\in P} \Theta^k_s \iff (M', s') \in P$$

Proof.

$\Longleftarrow$: Let $(M', s') \in P$.

For P union-closed; with the empty team property; invariant under k-bisimulation:

$$M^{'}; s^{'} \ (\ \underset{(M;s)\square P}{\square} \ \square\Theta^{k}_{s} \ \square\square \ (M^{'}; s^{'}) \ \square \ P$$

Proof.

$\square\square$: Let $(M^{'}; s^{'}) \ \square \ P$.

$s^{'} \ (\ \Theta^{k}_{s'}$, so $s^{'} \ (\ \square\Theta^{k}_{s'}$.

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Proof.

$\square\square$: Let $(M', s') \square P$.

$s' \; (\; \Theta^k_{s'}$, so $s' \; (\; \square\Theta^k_{s'}$.

$\square(M, s) \square P \; \square \; \square \; (\; \square\Theta^k_s$.

For P union-closed; with the empty team property; invariant under k-bisimulation:

$$M^{œ}; s^{œ} (\quad \underset{\hat{} M; s \bullet > P}{} \quad \wedge \quad \overset{k}{s} \qquad \hat{} M^{œ}; s^{œ} \bullet > P$$

[redacted]

Ô : Let $\hat{} M^{œ}; s^{œ} \bullet > P$.

$s^{œ} (\quad \overset{k}{s^{œ}}$ so $s^{œ} (\quad \wedge \quad \overset{k}{s^{œ}}$.

$¦ \hat{} M; s \bullet > P \quad g (\quad \wedge \quad \overset{k}{s}$.

$s^{œ} \quad s^{œ} 8 \, g$.

For P union-closed; with the empty team property; invariant under k-bisimulation:

M œ; s œ(^ k s ^M œ; s œ• > P

ˆM ; s• >P

[redacted]

Ô : Let ˆM œ; s œ• > P.

s œ(k sœ so s œ(^ k sœ•

¦ ˆM ; s• > P g (^ k s.

s œ s œ8 g .

Therefore

s œ(ˆM ; s• >P ^ k s.

For P union-closed; with the empty team property; invariant under k-bisimulation:

M œ;s œ(^M ;s• >P ^ k s ^M œ;s œ• > P

[redacted]

Ô : Assume s œ(M ;s>P ^ k s .

Ô : Let ^M œ;s œ• > P.

s œ(k s œ so s œ(^ k s œ.

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M œ; sœ (^ k s ^M œ; sœ • > P

^M ; s• >P

[redacted]

Ô : Let ^M œ; sœ • > P.

sœ (k sœ so sœ (^ k sœ •

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sœ sœ 8 g .

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Ô : Assume sœ (M ; s>P ^ k s .

Then sœ T where
¦ t > T t g or § ^Mt ; st • > P t (k st

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[redacted]

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¦ ^M ; s• > P g (^ k s .

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¦ t > T t g or § ^M t ; s t • > P t (k s t
¦ t > T t g or § ^M t ; s t • > P t - k s t

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[redacted]

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sœ($^{k}_{sœ}$ so sœ(^ $^{k}_{sœ}$.

¦ ˆM ; s• > P g (^ $^{k}_{s}$.

sœ sœ8 g .

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sœ($_{ˆM ; s• >P}$ ^ $^{k}_{s}$.

Ô : Assume sœ($_{M ; s>P}$ ^ $^{k}_{s}$.

Then sœ T where

¦ t > T t g or § ˆM_t ; s_t • > P t ($^{k}_{s_t}$

¦ t > T t g or § ˆM_t ; s_t • > P t - $_k$ s_t

If ¦ t t g , then sœ g > P by the empty team property.

For P union-closed; with the empty team property; invariant under k-bisimulation:

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ˆM ; s• > P

[redacted]

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If ¦ t t g , then sœ g > P by the empty team property.

Otherwise let T œ ˜t > T St x g• and consider M &˜Mt St > T œ• and its team u ˜st St > T œ•.

For P union-closed; with the empty team property; invariant under k-bisimulation:

$$M', s' \;(\; \square_{(M,s)\square P} \;\square\Theta^k_s \;\square\square\; (M', s') \,\square\, P$$

Proof.

$\square\square$: Let $(M', s') \,\square\, P$.

$s' \;(\; \Theta^k_{s'}$, so $s' \;(\; \square\Theta^k_{s'}$.

$\square(M,s) \,\square\, P \,\square\, \square \;(\; \square\Theta^k_s$.

$s' = s' \,\square\, \square$.

Therefore $s' \;(\; \square_{(M,s)\square P} \square\Theta^k_s$.

$\square\square$: Assume $s' \;(\; \square_{M,s\square P} \square\Theta^k_s$.

Then $s' = \square\, T$ where
$\square t \,\square\, T \,\square\, t = \square$ or $\square(M_t, s_t) \,\square\, P \,\square\, t \;(\; \Theta^k_{s_t}$
$\square t \,\square\, T \,\square\, t = \square$ or $\square(M_t, s_t) \,\square\, P \,\square\, t \;-_k\; s_t$

If $\square t \,\square\, t = \square$, then $s' = \square \,\square\, P$ by the empty team property.

Otherwise let $T' = \{t \,\square\, T \,\square\, t \neq \square\}$ and consider $M = \square\{M_t \,\square\, t \,\square\, T'\}$ and its team $u = \square\{s_t \,\square\, t \,\square\, T'\}$.
$(M, u) \,\square\, P$ by invariance and union closure.
And $s' \;-_k\; u$ so $s' \,\square\, P$. □

[redacted]

˜SSSS S> BSMLI•

˜P SP is invariant under k-bisimulation for some k > N•

Characteristic formulas: ˆM ;s• >P k s

[redacted]

˜SSSS S> BSMLE•

˜P SP is union closed and invariant under k-bisimulation for some k > N•

Characteristic formulas: ∧ ˆM ;s• >P k s ˆ ∧ ˆM ;s• >P k s • , ne

BSML^1 axiomatization

and : classical formulas (no ne or 1).

Non-modal portion (adapted from the system for PT):

introduction	elimination
D ‡ / – / I ˆ‡•	D$_1$ D$_2$ / E

ˆ‡• The undischarged assumptions in D ‡ do not contain ne.

, introduction

D_1 D_2

, I

,

, elimination

D

,

, E

D

,

, E

1 introduction

D

1 I

1

D

1 I

1

1 elimination

D

1

D_1 D_2

1 E

- weak introduction

D

———— - I ˆ‡‡•

-

- weakening

D

———— - W

-

- weak elimination

D D$_1^‡$ D$_2^‡$

-

———————————— - E ˆ‡; „•

- weak substitution

D D$_1^‡$

-

———————— - Sub ˆ‡•

-

ˆ‡• The undischarged assumptions in D$_1^‡$; D$_2^‡$ do not contain ne.

ˆ‡‡• may not contain ne.

ˆ„• may not contain 1 outside the scope of an .

- commutativity

D

-

Com-

-

- associativity

D

^ - • -

Ass-

- ^ - •

-1 distributivity

D

- ^ 1 •

Distr - 1

^ - • 1 ^ - •

– Elimination

D
- –
–E

á – , ne

á elimination

D
á
á E

á contraction

D
á -
á Ctr

ne introduction

$$\frac{}{-1\ \text{ne}}\ \text{neI}$$

- ne elimination

ˆ , ne• - ˆ , ne•

D D1 D2 D3

-

- neE

New rules for :

ne elimination

$$\dfrac{\begin{array}{c}D\\ \text{ne}\end{array}}{\_}\ \text{neE}$$

Double elimination

$$\dfrac{D}{\ }\ \text{DN}$$

De Morgan 1

$$\dfrac{\begin{array}{c}D\\ \hat{}\quad ,\quad \bullet\end{array}}{-}\ \text{DM}_1$$

De Morgan 2

$$\dfrac{\begin{array}{c}D\\ \hat{}\quad -\quad \bullet\end{array}}{,}\ \text{DM}_2$$

De Morgan 3

$$\dfrac{\begin{array}{c}D\\ \hat{}\quad 1\quad \bullet\end{array}}{,}\ \text{DM}_3$$

Modal portion|basic rules:

n monotonicity

D œ D

n

n n Mon ˆ‡•

j monotonicity

1 ::: n

D œ D1 Dn

j 1 ::: j n

j j Mon ˆ‡•

nj interaction

D

n

j Inter n j

ˆ‡• D œ does not contain undischarged assumptions.

New modal rules:

n 1 - conversion

D
n ˆ 1 •
――――――― Conv n 1-
n - n

j 1 - conversion

D
j ˆ 1 •
――――――― Conv j 1-
j - j

n separation

D

n ^ - ^ , ne••

n

n Sep

n join

D_1 D_2

n n

n ^ - •

n Join

j instantiation

D

j ^ , ne•

n

j Inst

jn join

D_1 D_2

j n

j ^ - •

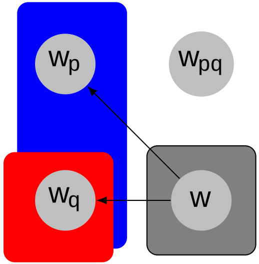
j n Join

s(n ¦ w >s §t b R w t x g and t (

s(j ¦ w >s R w (

D

n ˆ - ˆ , ne•• / n — n Sep



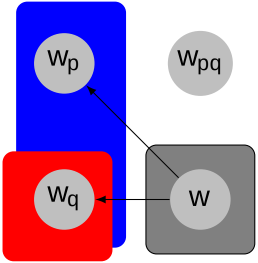
s(n ˆp - ˆq , ne••

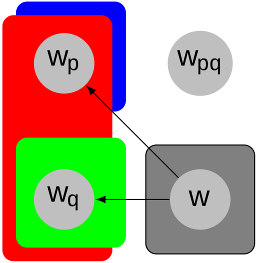
s(n q

s(n ¦ w >s §t b R w t x g and t (

s(j ¦ w >s R w (

$$\frac{\text{D} \quad n \,\hat{}\; - \;\hat{}\;\;,\; ne\bullet\bullet}{n}\; n\ \text{Sep} \qquad \frac{\text{D}_1 \; n \qquad \text{D}_2 \; n}{n \,\hat{}\; - \;\; \bullet}\; n\ \text{Join}$$





s (n ˆp - ˆq , ne•• s (n p , n q

s (n q s (n ˆp - q•

s (n ¦ w >s §t b R w t x g and t (

s (j ¦ w >s R w (

D

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n

D_1 D_2

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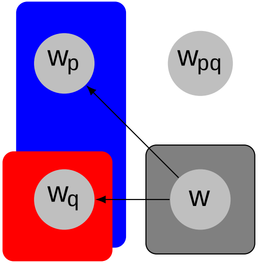
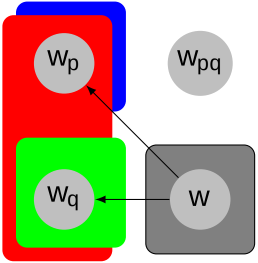
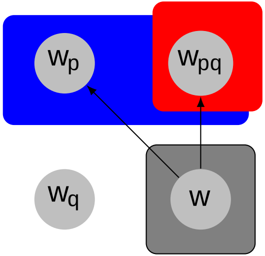
―――――――――― n Join
n ˆ - •

D_1 D_2

j n
―――――――――― j n Join
j ˆ - •







s(n ˆp - ˆq , ne••
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|\ought implies may" for pragmatically enriched formulas.

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

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Lemma: >BSML[1] Ô ¦ k C modal deptĥ • §P ÚØ k s ˆM ;s• >P

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(Ô $_{\hat{}M;s\bullet>P}$ ^k_s ($_{\hat{}N;t\bullet>Q}$ ^k_t

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(Ô $_{\text{ˆM;s•>P}}$ ^k_s ($_{\text{ˆN;t•>Q}}$ ^k_t

Ô ¦ ˆM;s• >P §ˆN;t• >Q s - $_k$ t

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Ô ¦ ^M;s• >P §^N;t• >Q s - k t
k s ÚØ k t

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(Ô ^M ;s• >P k s (^N ;t • >Q k t

Ô ¦ ^M ; s• >P §^N ; t • >Q s - k t k s ÚØ k t

Ô ^M ;s• >P k s Ø ^N ;t • >Q k t

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(Ô ˆM ;s• >P k s (ˆN ;t • >Q k t

Ô ¦ ˆM ; s• > P §ˆN ; t • > Q s - k t
k s ÚØ k t

Ô ˆM ;s• >P k s Ø ˆN ;t • >Q k t Ô Ø

BSML$^\wedge$ axiomatization

Exclude1 -rules and- neE; and add:

BSML^ axiomatization

Exclude1 -rules and- neE; and add:

^ 1 –

BSML^ BSML1

^ introduction	1 introduction
D / – / ^ I / ^ D / ^ I / ^	D / 1 I / 1 D / 1 I / 1

BSML^ axiomatization

Exclude1 -rules and- neE; and add:

^ 1 –

BSML^ BSML^1

^ introduction

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BSML$^{\wedge}$ axiomatization

Exclude1 -rules and- neE; and add:

$\wedge \quad 1 - \quad \wedge$

BSML$^{\wedge}$ | BSML^{1}

BSML$^{\wedge}$	BSML^{1}
$\wedge$ introduction: $\dfrac{D}{\wedge}\wedge I$, $\dfrac{D}{\wedge}\wedge I$	1 introduction: $\dfrac{D}{1}1I$, $\dfrac{D}{1}1I$
$\wedge$ ne introduction: $\dfrac{}{\wedge\,ne}\wedge neI$	ne introduction: $\dfrac{}{-1\ ne}neI$
$\wedge$ introduction: $\dfrac{D}{\wedge}\wedge\ I$	

BSML^	BSML[1]
^ elimination	1 elimination
ˆ–~ ^ ; m • ˆ ~ ^ ; m •	
D D_1 D_2	D D_1 D_2
	1
——— ^ Eˆ‡•	——— 1 E
ˆ‡• The occurrence at index m is not within the scope of or n .	

BSML^ BSML[1]

n^ elimination

ˆ–~ ^ ; m • ˆ ~ ^ ; m •

D D_1 D_2

n 1 2 n ^ Eˆ‡•

n 1 - n 2

n 1 - conversion

D

n ˆ 1 •

n - n Conv n 1-

ˆ‡• The occurrence at index m is not within the scope of a modality which occurs in , and not within the scope of (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

BSML^ BSML[1]

n^ elimination

ˆ–~ ^ ; m • ˆ ~ ^ ; m •

D D1 D2

n 1 2

n 1 - n 2 n ^ Eˆ‡•

j^ elimination

ˆ–~ ^ ; m • ˆ ~ ^ ; m •

D D1 D2

j 1 2

j 1 - j 2 j ^ Eˆ‡•

ˆ‡• The occurrence at index m is not within the scope of a modality which occurs in , and not within the scope of (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

n 1 - conversion

D

n ˆ 1 •

n - n Conv n 1-

j 1 - conversion

D

j ˆ 1 •

j - j Conv j 1-

Completeness

Lemma:

>BSMLE Ô ¦ k Cmď • §P ÚØ ^ k s or ÚØˆ ^ k s• , ne

ˆM ;s• >P ˆM ;s• >P

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ^ k s or ÚØˆ ^ k s • , ne

ˆM ;s• >P ˆM ;s• >P

(

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ^M ;s• >P ^ k s or ÚØˆ ^M ;s• >P ^ k s • , ne

(Ô ^M ;s• >P ^ k s (^N ;t • >Q ^ k t

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ˆM;s•>P ^ k s or ÚØˆ ˆM;s•>P ^ k s• , ne

(Ô ˆM;s•>P ^ k s (ˆN;t•>Q ^ k t

Ô ¦ ˆM;s• >P §R b Q s - k & R

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ^ k s ˆM ;s• >P or ÚØˆ ^ k s • , ne ˆM ;s• >P

(Ô ^ k s (^ k t
ˆM ;s• >P ˆN ;t • >Q

Ô ¦ ˆM ; s• > P §R b Q s - k & R

^ k t ˆN ;t • >Q RbQ k & R

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ^ k s or ÚØˆ ^ k s • , ne
ˆM ;s• >P ˆM ;s• >P

(Ô ^ k s (^ k t
ˆM ;s• >P ˆN ;t • >Q

Ô ¦ ˆM ; s• > P §R b Q s - k & R
k s Ø ^ k t
ˆN ;t • >R

^ k t k & R
ˆN ;t • >Q RbQ

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ˆM ;s• >P ^ k s or ÚØˆ ˆM ;s• >P ^ k s • , ne

(Ô ˆM ;s• >P ^ k s (ˆN ;t • >Q ^ k t

Ô ¦ ˆM ; s• >P §R b Q s - k & R

k s Ø ˆN ;t • >R ^ k t

k s Ø ˆN ;t • >Q ^ k t

ˆN ;t • >Q ^ k t RbQ k & R

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ^ k s or ÚØˆ ^ k s • , ne
ˆM ;s• >P ˆM ;s• >P

(Ô ^ k s (^ k t
ˆM ;s• >P ˆN;t • >Q

Ô ¦ ˆM ; s• > P §R b Q s - k & R
k s Ø ^ k t
ˆN;t • >R
k s Ø ^ k t
ˆN;t • >Q
^ k s Ø ^ k t
ˆN;t • >Q

^ k t k & R
ˆN;t • >Q RbQ

Completeness

Lemma:

>BSMLE Ô ¦ k Cmdˆ • §P ÚØ ^ k s or ÚØˆ ^ k s • , ne
ˆM ;s• >P ˆM ;s• >P

(Ô ^ k s (^ k t
ˆM ;s• >P ˆN ;t • >Q

Ô ¦ ˆM ; s• >P §R b Q s - k & R
k s Ø ^ k t
ˆN ;t • >R
k s Ø ^ k t
ˆN ;t • >Q
^ k s Ø ^ k t
ˆN ;t • >Q

^ k t k & R
ˆN ;t • >Q RbQ

Ô ^ k s Ø ^ k t
ˆM ;s• >P ˆN ;t • >Q

Completeness

Lemma:

>BSMLE Ô ¦ k Cmd ˆ • §P ÚØ ˆM ;s• >P ^ k s or ÚØˆ ˆM ;s• >P ^ k s • , ne

(Ô ˆM ;s• >P ^ k s (ˆN ;t • >Q ^ k t

Ô ¦ ˆM ; s• >P §R b Q s - k & R

k s Ø ˆN ;t • >R ^ k t

k s Ø ˆN ;t • >Q ^ k t

^ k s Ø ˆN ;t • >Q ^ k t

ˆN ;t • >Q ^ k t RbQ k & R

Ô ˆM ;s• >P ^ k s Ø ˆN ;t • >Q ^ k t Ô Ø

BSML axiomatization

Exclude1 -rules and- neE from BSML^1 and add:

BSML axiomatization

Exclude1 -rules and- neE from BSML[1] and add:

BSML

–ne translation

ˆ , – ~ ; m • ˆ , ne~ ; m •

D D₁ D₂

–neTrsˆ‡•

ˆ‡• The occurrence at indexm is not within the scope of or n .

BSML[1]

ne introduction

–1 ne neI

BSML$^\wedge$ BSML^1

n– ne translation

ˆ ,– ~ ; m • ˆ , ne~ ; m •

D D_1 D_2

n 1 2

n 1 - n 2 n– neTrsˆ‡•

n1- conversion

D

n ˆ 1 •

n - n Convn1-

ˆ‡• The occurrence at index m is not within the scope of a modality which occurs in , and not within the scope of (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

BSML^ BSML[1]

n– ne translation

ˆ ,– ~ ; m • ˆ , ne~ ; m •

D D1 D2
n 1 2
n 1 - n 2   n– neTrsˆ‡•

j– ne translation

ˆ ,– ~ ; m • ˆ , ne~ ; m •

D D1 D2
j 1 2
j 1 - j 2   j– neTrsˆ‡•

ˆ‡• The occurrence at indexm is not within the scope of a modality which occurs in , and not within the scope of (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

n 1 - conversion

D
n ˆ 1 •
n - n   Convn 1-

j 1 - conversion

D
j ˆ 1 •
j - j   Convj 1-

Old rules in [7]/[2] which are derivable:

k: set of all non-equivalent $^k_{s_i}$ over , where s_i x g
ne k

BSML BSML[1]

ne elimination	1 elimination
D; ^ $^k_{s_1}$ ~ ne; m • D₁ … ^ $^k_{s_n}$ ~ ne; m • D_n ⟹ neE ˆ‡• ˆ‡• The occurrence at index m is not within the scope of or n ; k > N; ~ s_1; : : : ; s_n • k.	D 1; D_1; D_2 ⟹ 1 E

BSML

n ne elimination

$$\frac{\begin{matrix} D \\ n \end{matrix} \quad \begin{matrix} \hat{\ } \; {}^{k}_{s_1} \sim ne; m \bullet \\ D_1 \\ {}_1 \end{matrix} \quad ::: \quad \begin{matrix} \hat{\ } \; {}^{k}_{s_n} \sim ne; m \bullet \\ D_n \\ {}_n \end{matrix}}{{}_{i>I}\, n \quad {}_i} \; n\, neE\hat{\ }\ddagger\bullet$$

j ne elimination

$$\frac{\begin{matrix} D \\ j \end{matrix} \quad \begin{matrix} \hat{\ } \; {}^{k}_{s_1} \sim ne; m \bullet \\ D_1 \\ {}_1 \end{matrix} \quad ::: \quad \begin{matrix} \hat{\ } \; {}^{k}_{s_n} \sim ne; m \bullet \\ D_n \\ {}_n \end{matrix}}{{}_{i>I}\, j \quad {}_i} \; j\, neE\hat{\ }\ddagger\bullet$$

ˆ‡• The occurrence at index m is not within the scope of a modality which occurs in , and not within the scope of (except if the forms part of j); $D_1; :::; D_n$ do not contain undischarged assumptions; $k > N$; ˜ $s_1; :::; s_n \bullet$ ^{k}.

BSML[1]

n 1 - conversion

$$\frac{\begin{matrix} D \\ n\, \hat{\ } \;\; 1 \quad \bullet \end{matrix}}{n \quad - n} \; Conv\, n\, 1\text{-}$$

j 1 - conversion

$$\frac{\begin{matrix} D \\ j\, \hat{\ } \;\; 1 \quad \bullet \end{matrix}}{j \quad - j} \; Conv\, j\, 1\text{-}$$

$$\frac{D \qquad \overset{\hat{}}{} \; {}^{k}_{s_1} \sim ne;m \bullet \; D_1 \quad \ldots \quad \overset{\hat{}}{} \; {}^{k}_{s_n} \sim ne;m \bullet \; D_n}{} \; neE\hat{}\ddagger\bullet$$

ˆ‡• The occurrence at index m is not within the scope of or n ; k > N; ˜ s_1; : : : ; s_n • ^{k}.

Let ${}^k_s$ $_{w>s}$ ${}^k_w$. By classical completeness Ø ${}^k_s$, where s is such that for each k th Hintikka formula ${}^k_{w}$œ, there is a w > s such that ${}^k_w$ ÚØ ${}^k_w$œ.

$$\frac{D \qquad \begin{matrix} \hat{}\ \ {}^{k}_{s_1} \sim ne; m \bullet \\ D_1 \end{matrix} \quad \dots \quad \begin{matrix} \hat{}\ \ {}^{k}_{s_n} \sim ne; m \bullet \\ D_n \end{matrix}}{} \; neE\hat{}\ddagger\bullet$$

ˆ‡• The occurrence at index m is not within the scope of or n ; k > N; ˜ $_{s_1}$; : : : ; $_{s_n}$• ^{k}.

Let $^{k}_{s}$ $_{w>s}$ $^{k}_{w}$. By classical completeness Ø $^{k}_{s}$, where s is such that for each k th Hintikka formula $^{k}_{w}$œ, there is a w > s such that $^{k}_{w}$ ÚØ $^{k}_{w}$œ.

Then it can be shown that Ø ˆ $^{k}_{s}$, ne ~ ne; m •. Let ˆ $^{k}_{s}$, ne ~ ne; m •.

$$\cfrac{D \qquad \overset{D_1}{\hat{\ } \ {}^{k}_{s_1} \sim ne;m \bullet} \qquad \vdots \qquad \overset{D_n}{\hat{\ } \ {}^{k}_{s_n} \sim ne;m \bullet}}{neE\hat{\ }\ddagger\bullet}$$

ˆ‡• The occurrence at index m is not within the scope of or n ; k > N; ˜ $_{s_1}$; : : : ; $_{s_n}$• ^{k}.

Let ^k_s $_{w>s}$ ^k_w. By classical completeness Ø ^k_s, where s is such that for each k th Hintikka formula $^k_{w_{œ}}$, there is a w > s such that ^k_w ÚØ $^k_{w_{œ}}$.

Then it can be shown that Ø ˆ ^k_s , ne~ ne; m • . Let ˆ ^k_s , ne~ ne; m • .

Consider the case in which S S 2. Let ^k_s $^k_{w_1}$ - $^k_{w_2}$.

ˆ $^{k}_{s_1}$ ~ ne; m • ˆ $^{k}_{s_n}$ ~ ne; m •

D D_1 ... D_n

neEˆ‡•

ˆ‡• The occurrence at index m is not within the scope of or n ; k > N; ˜ $_{s_1}$; : : : ; $_{s_n}$• ^{k}.

Let $^{k}_{s}$ $_{w>s}$ $^{k}_{w}$. By classical completeness Ø $^{k}_{s}$, where s is such that for each k th Hintikka formula $^{k}_{w}$œ, there is a w > s such that $^{k}_{w}$ ÚØ $^{k}_{w}$œ.

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Consider the case in which S 2. Let $^{k}_{s}$ $^{k}_{w_1}$ - $^{k}_{w_2}$.

Assume ˆ $^{k}_{w_1}$, – ~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, – ~ $^{k}_{w_2}$• for –neTrs. This is equivalent to ˆ– , ne~ ne; m • , which gives via the á-rules.

ˆ $^{k}_{s_1}$ ~ ne; m • ˆ $^{k}_{s_n}$ ~ ne; m •

D D_1 ... D_n

: : :

neEˆ‡•

ˆ‡• The occurrence at index m is not within the scope of or n ; k >N; ˜ $_{s_1}$; : : : ; $_{s_n}$ • ^{k}.

Let $^{k}_{s}$ $_{w>s}$ $^{k}_{w}$. By classical completeness Ø $^{k}_{s}$, where s is such that for each k th Hintikka formula $^{k}_{w}$ œ, there is a w > s such that $^{k}_{w}$ ÚØ $^{k}_{w}$ œ.

Then it can be shown that Ø ˆ $^{k}_{s}$, ne~ ne; m • . Let ˆ $^{k}_{s}$, ne~ ne; m • .

Consider the case in which S S 2. Let $^{k}_{s}$ $^{k}_{w_1}$ - $^{k}_{w_2}$.

Assume ˆ $^{k}_{w_1}$, – ~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, – ~ $^{k}_{w_2}$• for –neTrs. This is equivalent to ˆ–, ne~ ne; m •, which gives via the á-rules.

Assume ˆ $^{k}_{w_1}$, – ~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, ne~ $^{k}_{w_2}$• for –neTrs. This is equivalent to ˆ $^{k}_{w_2}$, ne~ ne; m •, which gives by assumption.

$$\frac{D \qquad \begin{matrix} \hat{}\ ^{k}_{s_1} \sim ne; m \bullet \\ D_1 \end{matrix} \qquad ::: \qquad \begin{matrix} \hat{}\ ^{k}_{s_n} \sim ne; m \bullet \\ D_n \end{matrix}}{} \ neE\hat{}\ddagger\bullet$$

ˆ‡• The occurrence at index m is not within the scope of or n ; $k > N$; ˜ $_{s_1}$; : : : ; $_{s_n}$• ^{k}.

Let $^{k}_{s}$ $_{w>s}$ $^{k}_{w}$. By classical completeness Ø $^{k}_{s}$, where s is such that for each k th Hintikka formula $^{k}_{w\,œ}$, there is a $w > s$ such that $^{k}_{w}$ ÚØ $^{k}_{w\,œ}$.

Then it can be shown that Ø ˆ $^{k}_{s}$, ne~ ne; m • . Let ˆ $^{k}_{s}$, ne~ ne; m • .

Consider the case in which S 2. Let $^{k}_{s}$ $^{k}_{w_1}$ - $^{k}_{w_2}$.

Assume ˆ $^{k}_{w_1}$, – ~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, – ~ $^{k}_{w_2}$• for –neTrs. This is equivalent to ˆ– , ne~ ne; m • , which gives via the á-rules.

Assume ˆ $^{k}_{w_1}$, – ~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, ne~ $^{k}_{w_2}$• for –neTrs. This is equivalent to ˆ $^{k}_{w_2}$, ne~ ne; m • , which gives by assumption.

Similarly ˆ $^{k}_{w_1}$, ne~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, – ~ $^{k}_{w_2}$• Ø and ˆ $^{k}_{w_1}$, ne~ $^{k}_{w_1}$•ˆ $^{k}_{w_2}$, ne~ $^{k}_{w_2}$• Ø .

So Ø by iterated applications of –neTrs.

Completeness

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation f for a formula , each atom is replaced by some 0 s such that s (:

p - ˆq , ne• Ô Ô f 0 sp - ˆ 0 sq , 0 ne •

Completeness

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation $_f$ for a formula , each atom is replaced by some ^0_s such that s (:

$$p - \hat{} q , ne \bullet \quad \hat{O} \quad \hat{O} \quad f \quad {}^0_{s_p} - \hat{} {}^0_{s_q} , {}^0_{ne} \bullet$$

F : the set of all instantiations of

Completeness

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$$p - \hat{} q , ne\bullet \quad \hat{O} \quad \hat{O} \quad f \quad {}^0_{s_p} - \hat{} \ {}^0_{s_q} , \ {}^0 ne \bullet$$

F : the set of all instantiations of

Since for each atom we have $\hat{} M ; s \bullet > P$ ^0_s, where P SSSS˜ˆ M ; s• SM ; s (•, then assuming that 1 distributes over everything:

$$F$$

And given rules that simulate 1 :

$$\text{¦} \ _f > F \quad _f \ Ø$$

$$\text{if ¦} \ _f > F \quad ; \ _f \ Ø \ , \text{then} \ ; \quad Ø$$

Problem: in BSML, 1 does not distribute over n . For instance n ˆp 1 q• * n p 1 n q.

Problem: in BSML, 1 does not distribute over n . For instance n ˆ p 1 q • * n p 1 n q.

Solution: we treat maximal modal subformulas as atoms: n j q Ô s^{0}_{p} , $s^{2}_{nj\ q}$.

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Lemma (1 -distributive form): >BSML implies ÚØ œ where œ does not contain n e within the scope of a n , and is in negation normal form

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Lemma (1 -distributive form): >BSML implies ÚØ œ where œ does not contain n e within the scope of a n , and is in negation normal form

An instantiation f of in 1 -distributive form:

- each n e is replaced by some s^0_{sf} where s_f (n e
- each >˜p; p; n ; j • is replaced by some s^k_{sf} $\underset{w>s_f}{}$ w^k (s_f (; k m d ˆ •)

Problem: in BSML, 1 does not distribute over n . For instance n ˆp 1 q• * n p 1 n q.

Solution: we treat maximal modal subformulas as atoms: p; n j q Ô s^0_p , $s^2_{nj\ q}$.

Lemma (1 -distributive form): >BSML implies ÚØ œ where œ does not contain n e within the scope of a n , and is in negation normal form

An instantiation $_f$ of in 1 -distributive form:

each n e is replaced by some $s^0_{s_f}$ where s_f (n e

each >˜p; p; n ; j • is replaced by some $s^k_{s_f}$ $_{w>s_f}$ ^k_w (s_f (; k md ˆ •)

$$F$$

$$¦ \ _f > F \quad _f \ Ø$$

if ¦ $_f$ > F ; $_f$ Ø , then ; Ø

(
Ô F (F

(

Ô F (F

¦ f ¦ k C md^ • §sf1; sf2

f ÚØ k sf1 - k sf2

(

Ô F (F

Ô ˆsf$_1$;sf$_2$•>A k sf$_1$ - k sf$_2$ (ˆsg$_1$;sg$_2$•>B k sg$_1$ - k sg$_2$

¦ f ¦ k Cmdˆ • §sf$_1$; sf$_2$
f ÚØ k sf$_1$ - k sf$_2$

(
Ô F (F

Ô ˆsf1;sf2•>A k sf1 - k sf2 (ˆsg1;sg2•>B k sg1 - k sg2

¦ f ¦ k Cmdˆ • §sf1; sf2
f ÚØ k sf1 - k sf2

k s1 - k s2 k s1>t
t bs2
¦ t b s2 k s1>t Ø k s1 - k s2
¦ t b s2 ; k s1>t Ø
Ô ; k s1 - k s2 Ø

(

Ô F (F

Ô ˆsf1;sf2•>A k sf1 - k sf2 (ˆsg1;sg2•>B k sg1 - k sg2

Ô ˆsf1;sf2•>A t bsf2 k sf1>t (ˆsg1;sg2•>B ubsg2 k sg1>u

¦ f ¦ k Cmdˆ • §sf1; sf2
f ÚØ k sf1 - k sf2

k s1 - k s2 t bs2 k s1>t
¦ t b s2 k s1>t Ø k s1 - k s2
¦ t b s2 ; k s1>t Ø
Ô ; k s1 - k s2 Ø

(

Ô F (F

Ô ˆsf1;sf2•>A k sf1 - k sf2 (ˆsg1;sg2•>B k sg1 - k sg2

Ô ˆsf1;sf2•>A t bsf2 k sf1>t (ˆsg1;sg2•>B ubsg2 k sg1>u

Ô ¦ ˆsf1; sf2• >A; t b sf2 §ˆsg1; sg2• >B; u b sg2

k sf1>t - k k sg1>u

¦ f ¦ k Cmdˆ • §sf1; sf2

f ÚØ k sf1 - k sf2

---

k s1 - k s2 t bs2 k s1>t

¦ t b s2 k s1>t Ø k s1 - k s2

¦ t b s2 ; k s1>t Ø

Ô ; k s1 - k s2 Ø

(

Ô F (F

Ô ˆsf1;sf2•>A k sf1 - k sf2 (ˆsg1;sg2•>B k sg1 - k sg2

Ô ˆsf1;sf2•>A t bsf2 k sf1>t (ˆsg1;sg2•>B ubsg2 k sg1>u

Ô ¦ ˆsf1; sf2• >A; t b sf2 §ˆsg1; sg2• >B; u b sg2
k sf1>t - k k sg1>u
k sf1>t ÚØ k sg1>u Ø k sg1 - k sg2 Ø g Ø

¦ f ¦ k Cmdˆ • §sf1; sf2
f ÚØ k sf1 - k sf2

k s1 - k s2 t bs2 k s1>t
¦ t b s2 k s1>t Ø k s1 - k s2
¦ t b s2 ; k s1>t Ø
Ô ; k s1 - k s2 Ø

(

Ô F (F

Ô ˆsf₁;sf₂•>A k sf₁ - k sf₂ (ˆsg₁;sg₂•>B k sg₁ - k sg₂

¦ f ¦ k Cmdˆ • §sf₁; sf₂
f ÚØ k sf₁ - k sf₂

Ô ˆsf₁;sf₂•>A t b sf₂ k sf₁>t (ˆsg₁;sg₂•>B u b sg₂ k sg₁>u

Ô ¦ ˆsf₁; sf₂• >A; t b sf₂ §ˆsg₁; sg₂• >B; u b sg₂ k sf₁>t - k k sg₁>u k sf₁>t ÚØ k sg₁>u Ø k sg₁ - k sg₂ Ø g Ø

k s₁ - k s₂ t b s₂ k s₁>t
¦ t b s₂ k s₁>t Ø k s₁ - k s₂
¦ t b s₂ ; k s₁>t Ø
Ô ; k s₁ - k s₂ Ø

Ô ¦ ˆsf₁; sf₂• >A k sf₁ - k sf₂ Ø

(
Ô F (F

Ô ˆsf1;sf2•>A k sf1 - k sf2 (ˆsg1;sg2•>B k sg1 - k sg2

¦ f ¦ k Cmdˆ • §sf1; sf2 f ÚØ k sf1 - k sf2

Ô ˆsf1;sf2•>A t bsf2 k sf1>t (ˆsg1;sg2•>B ubsg2 k sg1>u

Ô ¦ ˆsf1; sf2• >A; t b sf2 §ˆsg1; sg2• >B; u b sg2
k sf1>t - k k sg1>u
k sf1>t ÚØ k sg1>u Ø k sg1 - k sg2 Ø g Ø

Ô ¦ ˆsf1; sf2• >A k sf1 - k sf2 Ø
Ô ¦ f >F f Ø

k s1 - k s2 t bs2 k s1>t
¦ t b s2 k s1>t Ø k s1 - k s2
¦ t b s2 ; k s1>t Ø
Ô ; k s1 - k s2 Ø

(
Ô F (F

Ô k sf1 - k sf2 (k sg1 - k sg2
ˆsf1;sf2•>A ˆsg1;sg2•>B

Ô k sf1>t (k sg1>u
ˆsf1;sf2•>A t b sf2 ˆsg1;sg2•>B u b sg2

Ô ¦ ˆsf1; sf2• >A; t b sf2 §ˆsg1; sg2• >B; u b sg2
k sf1>t - k k sg1>u
k sf1>t ÚØ k sg1>u Ø k sg1 - k sg2 Ø g Ø

Ô ¦ ˆsf1; sf2• >A k sf1 - k sf2 Ø
Ô ¦ f >F f Ø
Ô Ø

¦ f ¦ k Cmdˆ • §sf1; sf2
f ÚØ k sf1 - k sf2

k s1 - k s2 k s1>t
t b s2
¦ t b s2 k s1>t Ø k s1 - k s2
¦ t b s2 ; k s1>t Ø
Ô ; k s1 - k s2 Ø

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