

Modal team logics for modelling Free Choice inference

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Helsinki Logic Seminar

Overview

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Motivation: Free Choice (fC)

Aloni: Bilateral state-based modal logic ($BSML$) accounts for fC

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Motivation: Free Choice (fC)

Aloni: Bilateral state-based modal logic ($BSML$) accounts for fC

$BSML$ is not expressively complete. The following extensions are:

$BSML$: $BSML$ with the global (inquisitive) disjunction

$BSML$: $BSML$ with an "emptiness" operator

$BSML < BSML < BSML =$ Modal Team Logic (MTL)

Natural deduction axiomatizations

Free choice (fc) inference

Free choice (fc) inference

You may have coffee or tea.

You may have coffee and you may have tea.

(You may have both coffee and tea.)

A possible formalization:

(†) ()

Problem:

- | | | |
|----|----------------|----------------------------|
| 1: | p | |
| 2: | $(p \wedge q)$ | (1, classical modal logic) |
| 3: | q | (2, †) |

Bilateral State-based Modal Logic

Team semantics for modal logic

$$M = (W; R; V)$$

standard Kripke semantics

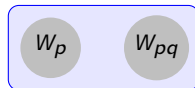
state-based/team semantics

$$M; w ($$

$$M; s ($$

$$w \quad W$$

$$s \quad W$$



$$w_p (p$$

$$\{w_p; w_{pq}\} (p$$

Bilateralism

“ is assertable in s ”

s (

“ is rejectable in s ”

)

Bilateralism

“ is assertable in s ” $s ($
 “ is rejectable in s ” $s)$

Bilateral negation

$s (\neg$ $s)$
 $s) \neg$ $s ($

Syntax of $BSML$:

$$= p \quad \neg \quad (\quad) \quad (\quad) \quad \text{ne}$$

Syntax of $BSML$:

$$= p \quad \neg \quad (\quad) \quad (\quad) \quad \text{ne}$$

Semantics $((\quad))$

$$\begin{aligned} s \langle p \rangle &= \{w \mid s \models w \models V(p)\} \\ s \langle \neg \phi \rangle &= \{w \mid s \not\models w \models \phi\} \\ s \langle \phi \wedge \psi \rangle &= \{w \mid s \models w \models \phi \text{ and } s \models w \models \psi\} \\ s \langle \exists x \phi \rangle &= \{w \mid \exists t \text{ such that } t \models \phi \text{ and } t \models w \models \phi\} \\ s \langle \forall x \phi \rangle &= \{w \mid \forall t \text{ such that } t \models w \models \phi \text{ and } t \models w \models \phi\} \\ s \langle \text{ne} \rangle &= \{w \mid s \not\models w \models \text{ne}\} \end{aligned}$$

$$R[w] = \{v \mid W \models wRv\}$$

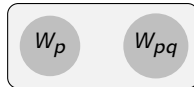
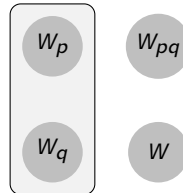
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Semantics (())

$$\begin{array}{l}
 s (p \\
 s (\neg \\
 s (\\
 s (\\
 s (\\
 s (\\
 s (\text{ne}
 \end{array}
 \quad
 \begin{array}{l}
 w \quad s \quad w \quad V(p) \\
 s) \\
 s (\quad \text{and } s (\\
 t; t \quad t \quad t = s \text{ and } t (\quad \text{and } t (\\
 w \quad s \quad t \quad R[w] \quad t \quad \text{and } t (\\
 s
 \end{array}$$

$$R[w] = \{v \mid W \quad wRv\}$$

$s(p)$ (a) $s(p)$ $w s w V(p)$ (b) $s * p$

Syntax of $BSML$:

$$= p \quad \neg \quad (\quad) \quad (\quad) \quad \text{ne}$$

Semantics $((\quad))$

$$\begin{array}{l}
 s \langle p \rangle \quad w \quad s \quad w \quad V(p) \\
 s \langle \neg \rangle \quad s \rangle \\
 s \langle \quad \rangle \quad s \langle \quad \text{and } s \langle \quad \rangle \\
 s \langle \quad \rangle \quad t; t \quad t \quad t = s \text{ and } t \langle \quad \rangle \quad \text{and } t \langle \quad \rangle \\
 s \langle \quad \rangle \quad w \quad s \quad t \quad R[w] \quad t \quad \text{and } t \langle \quad \rangle \\
 s \langle \text{ne} \rangle \quad s
 \end{array}$$

$$R[w] = \{v \mid W \quad wRv\}$$

Syntax of *BSML*:

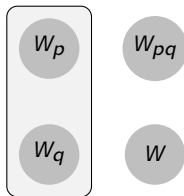
$$= p \quad \neg \quad (\quad) \quad (\quad) \quad \text{ne}$$

Semantics (())

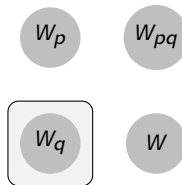
$$\begin{array}{l}
 s (p \quad \quad \quad w \quad s \quad w \quad V(p) \\
 s (\neg \quad \quad \quad s) \\
 s (\quad \quad \quad s (\quad \text{and } s (\\
 s (\quad \quad \quad t; t \quad t \quad t = s \text{ and } t (\quad \text{and } t (\\
 s (\quad \quad \quad w \quad s \quad t \quad R[w] \quad t \quad \text{and } t (\\
 s (\text{ne} \quad \quad \quad s
 \end{array}$$

$$R[w] = \{v \mid W \quad wRv\}$$

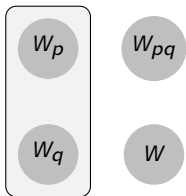
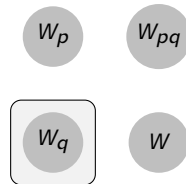
Tensor disjunction

 $s ($
 $t; t$
 $t \quad t = s \quad \text{and}$
 $t (\quad \text{and}$
 $t ($


(a) $s (p \quad q$



(b) $s (p \quad q$

The non-emptiness atom ne $s \ (\ ne$ s (a) $s \ (\ (p \ ne) \ (q \ ne)$ (b) $s^* \ (\ (p \ ne) \ (q \ ne)$

The modality

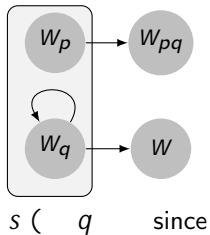
$$R[w] = \{v \mid W wRv\}$$

$s \models \Box \phi$ iff for all t such that $w R t$, $t \models \phi$ and $t \models \Box \phi$

The modality

$$R[w] = \{v \mid W wRv\}$$

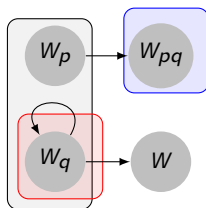
$s \models \Box p$ $w \models s \models t \in R[w]$ $t \models p$ and $t \models \Box p$



The modality

$$R[w] = \{v \mid W w R v\}$$

$s \models w$ and $t \models R[w]$ and $t \models w$



$s \models w$ since

$$\{w_q\} \models R[w_q] \quad \text{and} \quad \{w_{pq}\} \models R[w_p]$$

$$\{w_q\} \models w \quad \text{and} \quad \{w_{pq}\} \models w$$

Syntax of *BSML*:

$$= p \quad \neg \quad (\quad) \quad (\quad) \quad \text{ne}$$

Semantics (())

$$\begin{array}{l}
 s (p \quad \quad \quad w \quad s \quad w \quad V(p) \\
 s (\neg \quad \quad \quad s) \\
 s (\quad \quad \quad s (\quad \text{and } s (\\
 s (\quad \quad \quad t; t \quad t \quad t = s \text{ and } t (\quad \text{and } t (\\
 s (\quad \quad \quad w \quad s \quad t \quad R[w] \quad t \quad \text{and } t (\\
 s (\text{ne} \quad \quad \quad s
 \end{array}$$

$$R[w] = \{v \quad W \quad wRv\}$$

Accounting for fc

The empty team supports contradictions such as $p \ \neg p$

ne represents lack of contradiction

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fC is caused by an intrusion of the pragmatic principle “avoid stating a contradiction” (ne) into meaning composition:

Accounting for fC

The empty team supports contradictions such as $p \quad \neg p$

ne represents lack of contradiction

fC is caused by an intrusion of the pragmatic principle “avoid stating a contradiction” (ne) into meaning composition:

$$\begin{array}{lcl}
 p^+ & = & p \quad \text{ne} \\
 (\neg \quad)^+ & = & \neg \quad^+ \quad \text{ne} \\
 (\quad)^+ & = & (\quad^+ \quad^+) \quad \text{ne} \\
 (\quad)^+ & = & (\quad^+ \quad^+) \quad \text{ne} \\
 (\quad)^+ & = & \quad^+ \quad \text{ne}
 \end{array}$$

You may have coffee or tea.

You may have coffee and you may have tea.

$$(c \vee t)^+ (c \wedge t)$$

You may have coffee or tea.

You may have coffee and you may have tea.

$$(c \vee t)^+ \wedge (c \wedge t)$$

i.e. $((c \wedge \neg t) \vee (t \wedge \neg c)) \wedge (c \wedge t)$

You may have coffee or tea.

You may have coffee and you may have tea.

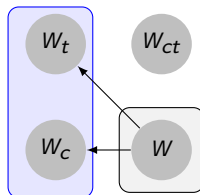
$$(c \vee t)^+ (c \wedge t)$$

i.e. $((c \wedge \neg e) \vee (t \wedge \neg e)) \wedge \neg e \wedge (c \wedge t)$

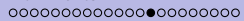
i.e. $((c \wedge \neg e) \vee (t \wedge \neg e)) \wedge (c \wedge t)$

$((c \text{ ne}) (t \text{ ne}))$
 $(c \text{ t})$

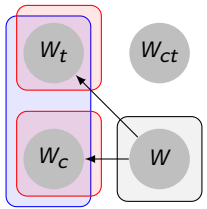
$((c \text{ ne}) (t \text{ ne}))$ (c t



$\{w\}$ ($((c \text{ ne}) (t \text{ ne}))$ since

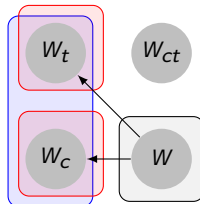


$$((c \text{ ne}) (t \text{ ne})) \quad (\quad c \quad t$$



$\{w\} (((c \text{ ne}) (t \text{ ne}))$ since $\{W_c\} (c$ and $\{W_t\} (t$

$((c \text{ ne}) (t \text{ ne})) \quad (c \text{ } t)$



$\{w\} \not\models ((c \text{ ne}) (t \text{ ne}))$ since $\{w_c\} \models c$ and $\{w_t\} \models t$

for the same reason, $\{w\} \not\models c \text{ } t$

$BSML$: $BSML$ with the global disjunction

$s (\quad \quad \quad)$ or $s (\quad \quad \quad)$

$BSML$: $BSML$ with the **global disjunction**

$s (\quad \quad \quad)$ or $s (\quad \quad \quad)$

$BSML$: $BSML$ with the **emptiness/circle operator**

$s (\quad \quad \quad)$ or $s =$

$BSML$: $BSML$ with the **global disjunction**

$$s (\quad \quad \quad) \quad \quad \quad s (\quad \quad \quad) \quad \text{or} \quad s (\quad \quad \quad)$$

$BSML$: $BSML$ with the **emptiness/circle operator**

$$s (\quad \quad \quad) \quad \quad \quad s (\quad \quad \quad) \quad \text{or} \quad s =$$

For classical formulas (no \neg ; \rightarrow ; \wedge):

$$s (\quad \quad \quad) \quad \quad \quad W \quad S \quad \{W\} (\quad \quad \quad) \quad \quad \quad W \quad S \quad W (\quad \quad \quad)$$

Semantics ())

s) p	$w \ s \ w \ V(p)$
s) \neg	s (
s)	$t; t \ t \ t = s \text{ and } t) \ \text{and } t)$
s)	s) and s)
s)	s) and s)
s)	$w \ s \ R[w])$
s) ne	s =
s)	s)

Semantics ())

s) p	w s w V(p)
s) ¬	s (
s)	t; t t t = s and t) and t)
s)	s) and s)
s)	s) and s)
s)	w s R[w])
s) ne	s =
s)	s)

= ¬ ¬

s (w s R[w] (

You may not have coffee or tea.

You may not have coffee and you may not have tea.

$$\neg (c \vee t) \quad (\neg c \wedge \neg t)$$

You may not have coffee or tea.

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$$\neg (c \vee t) \quad (\neg c \wedge \neg t)$$

In $BSML$: $(\neg (b \vee c))^+ (\neg b \wedge \neg c)$.

For classical : (ne)

Using we can define a function which cancels pragmatic enrichment:

$$\begin{array}{lcl}
 p^- & = & p \\
 ne^- & = & ne \\
 (\neg)^- & = & \neg^- \\
 (\)^- & = & \quad \quad - \\
 (\)^- & = & \quad \quad - \\
 (\)^- & = & \quad \quad -
 \end{array}$$

For classical (+)^-

Closure properties

is *downward closed*:

$$[M; s \models \varphi \text{ and } t \sqsubseteq s] \implies M; t \models \varphi$$

is *union closed*:

$$[M; s \models \varphi \text{ for all } s \in S] \implies M; \bigcup S \models \varphi$$

has the *empty team property*

$$M; \emptyset \models \varphi \text{ for all } M$$

is *at*

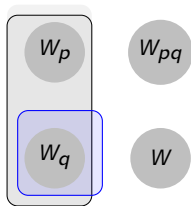
$$M; s \models \varphi \implies M; \{w\} \models \varphi \text{ for all } w \sqsubseteq s$$

flat downward closed & union closed & empty team property

Classical formulas are flat

Classical formulas are flat

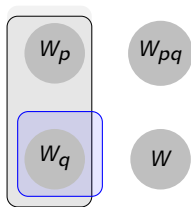
Formulas with ne may lack downward closure and the empty team property:



$$\begin{array}{l} \{W_p; W_q\} \\ \{W_q\} \end{array} \quad \begin{array}{l} (\\ * \end{array} \quad \begin{array}{l} (p \text{ ne}) \\ (p \text{ ne}) \end{array} \quad \begin{array}{l} (q \text{ ne}) \\ (q \text{ ne}) \end{array}$$

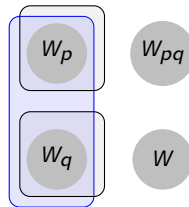
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Formulas with \cup may lack union closure:



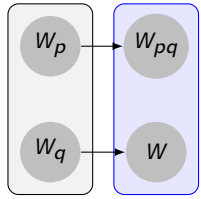
$$\begin{array}{l} \{W_p\} \\ \{W_q\} \\ \{W_p; W_q\} \end{array} \quad \begin{array}{l} (\\ (\\ * \end{array} \quad \begin{array}{l} p \ q \\ p \ q \\ p \ q \end{array}$$

The modal dependence logic modalities and

t is a successor team of s

sRt $\iff t \subseteq R[s]$ and $R[w] \subseteq t$ for all $w \in s$

$$R[s] = \{v \mid \exists w \subseteq s \exists v (wRv)\}$$

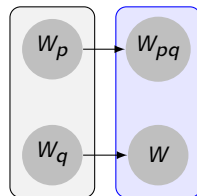


The modal dependence logic modalities and

t is a successor team of s

sRt iff $t \neq \emptyset$ and $R[s] \cap R[w] = \emptyset$ for all $w \in s$

$R[s] = \{v \mid \exists w \in s \exists v (wRv)\}$



$s \models \Box \varphi$ iff $t \models \varphi$ and $t \in R[s]$

$s \models \Box \varphi$ iff $R[s] \models \varphi$

$s \models \Box \varphi$ iff $\forall w \in s \exists t \in R[w] (t \models \varphi)$ and $t \in R[s]$

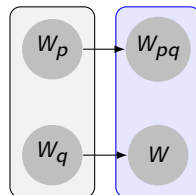
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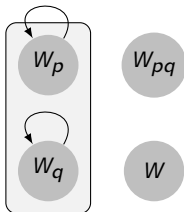
$s \models \Box \varphi$ iff $\forall w \in s R[w] \models \varphi$

If \Box is downward closed, $\Box \Box \varphi \equiv \Box \varphi$ and $\Box \Box \varphi \equiv \Box \varphi$

If \Box is union closed and has the empty team property, $\Box \Box \varphi \equiv \Box \varphi$ and $\Box \Box \varphi \equiv \Box \varphi$

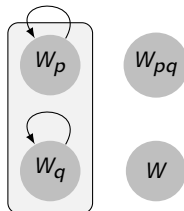
If \Box is flat, $\Box \Box \varphi \equiv \Box \varphi$ and $\Box \Box \varphi \equiv \Box \varphi$

Aloni's free choice explanation does not work with :



sRs and $s((p \text{ ne}) (q \text{ ne}))$
 Therefore $s((p \text{ ne}) (q \text{ ne}))$

Aloni's free choice explanation does not work with :



sRs and $s \langle (p \text{ ne}) \ (q \text{ ne}) \rangle$
 Therefore $s \langle \langle (p \text{ ne}) \ (q \text{ ne}) \rangle \rangle$

s is the the only successor team of s and $s \ast p$
 Therefore $s \ast p$ so $\langle \langle (p \text{ ne}) \ (q \text{ ne}) \rangle \rangle \ast p \ q$

Expressive Power

Fix a finite set of proposition symbols Φ

Pointed team model: $(M; s)$ where M is a model over Φ ; s is a team on M

Team property: set of pointed team models

$$= \{(M; s) \mid M; s \models \varphi\}$$

Expressive Power

Fix a finite set of proposition symbols Φ

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Theorem

$$\{ \varphi \mid \varphi \text{ is } BSML \text{ axiomatizable} \} = \{ \text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N} \}$$

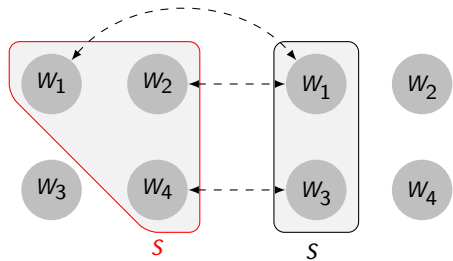
Team bisimulation:

$$S -_k S$$

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$$S -_k S$$

forth: $W \ S \ W \ S \ W -_k W$

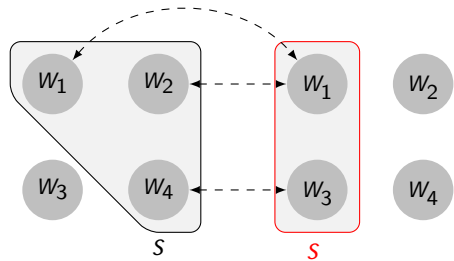


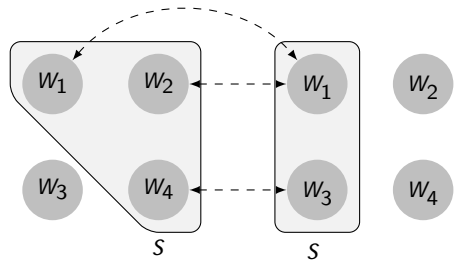
Team bisimulation:

 $S \sim_k S$

forth: $W \quad S \quad W \quad S \quad W \sim_k W$

back: $W \quad S \quad W \quad S \quad W \sim_k W$



Team bisimulation: $S -_k S$ forth: $W \quad S \quad W \quad S \quad W -_k W$ back: $W \quad S \quad W \quad S \quad W -_k W$ $S - S$ $k \quad N \quad S -_k S$ 

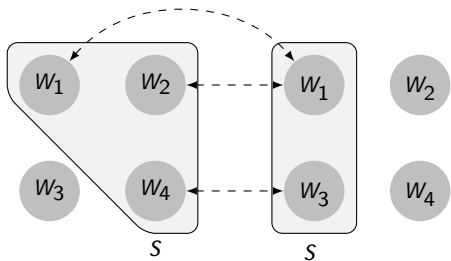
Team bisimulation:

$$S -_k S$$

forth: $W \quad S \quad W \quad S \quad W -_k W$

back: $W \quad S \quad W \quad S \quad W -_k W$

$$S - S \quad k \quad N \quad S -_k S$$



Theorem (bisimulation invariance)

$$S -_k S$$

$$S - S$$

$$S \stackrel{k}{=} S$$

$$S \quad S$$

Property P is *invariant under team k -bisimulation*:

$$\left[((M;s) \models P \text{ and } M;s \dot{-}_k M;s) \Rightarrow (M;s) \models P \right]$$

Theorem

$$\{ \text{BSML} \} = \{ \text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N} \}$$

Characteristic formulas for worlds

(Hintikka formulas):

$$\overset{0}{M;w} = \{p \ w \ V(p)\} \quad \{-p \ w \ V(p)\} \quad (p \ \Phi)$$

Characteristic formulas for worlds

(Hintikka formulas):

$$M;w^0 = \{p \mid w \Vdash V(p)\} \quad \{\neg p \mid w \Vdash V(p)\} \quad (p \mid \Phi)$$

$$M;w^{k+1} = \bigwedge_{v \in R[w]} M;v^k \quad \bigwedge_{v \in R[w]} M;v^k$$

Characteristic formulas for worlds

(Hintikka formulas):

$$M;w^0 = \{p \mid w \Vdash p\} \cap \{\neg p \mid w \Vdash \neg p\} \cap \{p \mid \Phi\}$$

$$M;w^{k+1} = \bigwedge_{v \in R[w]} M;v^k \cap \bigwedge_{v \in R[w]} M;v^k$$

$$W \left(\begin{array}{l} k \\ w \end{array} \right) \quad W - k \quad W$$

Characteristic formulas for **worlds** (Hintikka formulas):

$$\Theta_{M;w}^0 = \{p \mid w \models V(p)\} \cup \{\neg p \mid w \models V(p)\} \cup \{p \mid \Phi\}$$

$$\Theta_{M;w}^{k+1} = \bigwedge_{v \in R[w]} \Theta_{M;v}^k \quad \bigwedge_{v \in R[w]} \Theta_{M;v}^k$$

$$W \left(\begin{matrix} k \\ w \end{matrix} \quad W - k \quad W \right)$$

Characteristic formulas for **teams**:

$$\Theta_{M;s}^k = \begin{cases} \bigwedge_{w \in s} \Theta_{M;w}^k & \text{if } s = \{w \mid w \models p \wedge \neg p\} \\ \bigwedge_{w \in s} \Theta_{M;w}^k & \text{if } s \text{ is consistent} \end{cases}$$

$$\Theta_s^k = \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$$

$$s \left(\Theta_s^k \right) = s - k S$$

$$\Theta_s^k = \binom{k}{w \ s} \text{ ne}$$

$$s \left(\Theta_s^k \quad s - k \ S \right)$$

Proof.

Case 1 : $s = \dots$

Then $\Theta_s^k = \dots$ and

$s - k \ S \quad s = \dots \quad s \left(\dots \right)$

$$\Theta_s^k = \left(\begin{array}{c} k \\ w \end{array} \text{ ne} \right)$$

$$s \left(\Theta_s^k \quad s - k S \right)$$

Proof.

Case 1 : $s = \dots$

Then $\Theta_s^k = \dots$ and

$s - k S \quad s = \dots \quad s \left(\dots \right)$

Case 2: $s \dots$

:

Let $w \dots s$. Let $w \dots s$ be s.t. $w - k w$.

$$\Theta_s^k = \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$$

$$s \left(\Theta_s^k \quad s - k S \right)$$

Proof.

Case 1 : $s = \dots$

Then $\Theta_s^k = \dots$ and

$$s - k S \quad s = \dots \quad s \left(\dots \right)$$

Case 2: $s \dots$

:

Let $w \leq s$. Let $w \leq s$ be s.t. $w - k w$.

Then $w \left(\begin{matrix} k \\ w \end{matrix} \right)$

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \in s}$$

$$s \in \Theta_s^k \iff s - k \subseteq s$$

Proof.

Case 1 : $s = \emptyset$.

Then $\Theta_s^k = \emptyset$ and

$$s - k \subseteq s \iff \emptyset = \emptyset \iff \emptyset \in \emptyset.$$

Case 2: $s \neq \emptyset$.

:

Let $w \in s$. Let $w' \in s$ be s.t. $w' - k \subseteq w'$.

Then $w' \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \in s}$

so $\{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \in s}$ and $\{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \in s}$.

So $w' \in s \implies \{w'\} \in s \implies \{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \in s}$.

And $\{w'\} \in s \implies w' \in s \implies \{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \in s}$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$$

$$s \in \Theta_s^k \iff s - k \leq s$$

Proof.

Case 1 : $s = k$.

Then $\Theta_s^k = \left(\binom{k}{s} \text{ ne} \right)_{w \leq s}$ and

$$s - k \leq s \iff s = k \iff s \in \Theta_s^k.$$

Case 2: $s < k$.

:

Let $w \leq s$. Let $w' \leq s$ be s.t. $w' - k \leq w'$.

Then $w' \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \leq s}$

so $\{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \leq s}$ and $\{w'\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.

So $w' \leq s \implies \{w'\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.

And $\{w'\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s} \implies w' \leq s \implies \{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)_{w' \leq s}$.

Therefore $s \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$, i.e. $s \in \Theta_s^k$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \in s}$$

$$s \in \Theta_s^k \iff s - k \subseteq s$$

Proof.

Case 1 : $s = \emptyset$.

Then $\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \in s}$

$s - k \subseteq s$ iff $s = \emptyset$.

:

$s \in \Theta_s^k$ implies there are $s_w \subseteq s$ ($w \in s$) s.t. $s_w \left(\binom{k}{w} \text{ ne} \right)$ and $s = \bigcup_{w \in s} s_w$.

Case 2: $s \neq \emptyset$.

:

Let $w \in s$. Let $w' \subseteq s$ be s.t. $w' - k \subseteq w'$.

Then $w' \left(\binom{k}{w'} \text{ ne} \right)$

so $\{w'\} \left(\binom{k}{w'} \text{ ne} \right)$ and $\{w'\} \left(\binom{k}{w} \text{ ne} \right)$.

So $w' \subseteq s \implies \{w'\} \subseteq s \implies \{w'\} \left(\binom{k}{w} \text{ ne} \right)$.

And $\{w'\} \subseteq s \implies w' \subseteq s \implies \{w'\} \left(\binom{k}{w} \text{ ne} \right)$.

Therefore $s \left(\left(\binom{k}{w} \text{ ne} \right)_{w \in s} \right)$, i.e. $s \in \Theta_s^k$.

$$\Theta_s^k = \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)_{w \ s}$$

$$s \in \Theta_s^k \iff s \in \Theta_s^k$$

Proof.

Case 1 : $s = \cdot$.

Then $\Theta_s^k = \cdot$ and $s \in \Theta_s^k$.

$s \in \Theta_s^k \iff s = \cdot$.

:

$s \in \Theta_s^k$ implies there are $s_w \in \Theta_s^k$ s.t.

$s_w \in \Theta_s^k$ and $s = \cdot s_w$.

Case 2: $s = \cdot$.

:

Let $w \in s$. Let $w \in s$ be s.t. $w \in W$.

Then $w \in \Theta_s^k$ and $\{w\} \in \Theta_s^k$ ne.

So $w \in s$ and $\{w\} \in s$ and $\{w\} \in \Theta_s^k$ ne.

And $\{w\} \in s$ and $w \in s$ and $\{w\} \in \Theta_s^k$ ne.

Therefore $s \in \Theta_s^k$, i.e. $s \in \Theta_s^k$.

Want to show:

$$W \in s \iff W \in s \iff W \in W$$

$$W \in s \iff W \in s \iff W \in W.$$

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right) \quad s \in \Theta_s^k \iff s \in_{k-1} S$$

Proof.

Case 1 : $s = \dots$
 Then $\Theta_s^k = \dots$ and $s \in_{k-1} S$
 $s \in_{k-1} S \iff s = \dots$
 $s \in \Theta_s^k$ implies there are $s_w \in \binom{k}{w} \text{ ne}$ and $s = \dots$

Case 2: $s = \dots$
 Want to show:
 \dots

Let $w \in s$. Let $w \in s$ be s.t. $w \in_{k-1} W$.
 Then $w \in \binom{k}{w} \text{ ne}$ and $\{w\} \in \binom{k}{w} \text{ ne}$.
 Let $w \in s$. Then for some $w \in s$ we have $w \in S_w$.

So $w \in s \iff \{w\} \in s \iff \{w\} \in \binom{k}{w} \text{ ne}$.
 And $\{w\} \in s \iff w \in s \iff \{w\} \in \binom{k}{w} \text{ ne}$.
 Therefore $s \in \left(\binom{k}{w} \text{ ne} \right)$, i.e. $s \in \Theta_s^k$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$$

$$s \in \Theta_s^k \iff s - k \leq s$$

Proof.

Case 1 : $s = \dots$

Then $\Theta_s^k = \dots$ and

$$s - k \leq s \iff s \in \Theta_s^k$$

Case 2 : $s = \dots$

$s \in \Theta_s^k$ implies there are $s_w \leq s - k$ s.t. $s_w \in \Theta_{s_w}^k$ and $s = s_w + k$.

Case 2: $s = \dots$

Want to show:

$$\binom{k}{w} \text{ ne} \iff \binom{k}{w-k} \text{ ne} \iff \binom{k}{w} \text{ ne}$$

Let $w \leq s$. Let $w' \leq s$ be s.t. $w' - k \leq w'$.

Let $w \leq s$. Then for some $w' \leq s$ we have $w = w' + k$. Since $w' \in \Theta_{w'}^k$, we have $w' \in \Theta_{w'}^k$.

Then $w' \in \Theta_{w'}^k$ and $\{w\} \in \Theta_w^k$ ne.

So $w \leq s \implies \{w\} \in \Theta_w^k$ ne.
 And $\{w\} \in \Theta_w^k \implies w \leq s$ and $\{w\} \in \Theta_w^k$ ne.
 Therefore $s \in \Theta_s^k \iff \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$, i.e. $s \in \Theta_s^k$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$$

$$s \in \Theta_s^k \iff s - k \leq s$$

Proof.

Case 1 : $s = \dots$

Then $\Theta_s^k = \dots$
 $s - k \leq s \iff s = \dots$

:

$s \in \Theta_s^k$ implies there are $s_w \leq s$ ($w \leq s$) s.t.
 $s_w \in \left(\binom{k}{w} \text{ ne} \right)$ and $s = \sum_{w \leq s} s_w$.

Case 2: $s = \dots$

:

Let $w \leq s$. Let $w' \leq s$ be s.t. $w' - k \leq w'$.

Then $w' \in \left(\binom{k}{w'} \text{ ne} \right)$
 so $\{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)$ and $\{w'\} \in \left(\binom{k}{w'} \text{ ne} \right)$.

Want to show:

$$\sum_{w \leq s} s_w \in \left(\binom{k}{w} \text{ ne} \right) \iff \sum_{w \leq s} s_w \in \left(\binom{k}{w} \text{ ne} \right)$$

Let $w \leq s$. Then for some $w' \leq s$ we have
 $w' \leq s_w$. Since $s_w \in \left(\binom{k}{w'} \text{ ne} \right)$, we have $s_w \in \left(\binom{k}{w'} \text{ ne} \right)$.
 So $w \in \left(\binom{k}{w} \text{ ne} \right)$ whence $w - k \leq w$.

So $w \leq s \implies \{w\} \in \left(\binom{k}{w} \text{ ne} \right)$.

And $\{w\} \in \left(\binom{k}{w} \text{ ne} \right) \implies w \leq s$.

Therefore $s \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$, i.e. $s \in \Theta_s^k$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right) \quad s \in \Theta_s^k \iff s \in \Theta_{s-k}^k$$

Proof.

Case 1 : $s = \emptyset$.
 Then $\Theta_s^k = \emptyset$ and $s - k = \emptyset$.
 $s \in \Theta_s^k \iff \emptyset \in \Theta_{\emptyset}^k$ implies there are $s_w = \emptyset$ s.t. $s_w \in \Theta_{s-w}^k$ and $s = \sum_{w \in s} s_w$.

Case 2: $s \neq \emptyset$.
 Want to show:
 $\{w \in s\} \in \Theta_{s-w}^k$ iff $\{w \in s\} \in \Theta_s^k$.

Let $w \in s$. Let $s_w = \emptyset$ be s.t. $s_w - k = \emptyset$.
 Then $w \in \Theta_{s-w}^k$ and $\{w\} \in \Theta_{s-w}^k$.
 so $\{w\} \in \Theta_w^k$ and $\{w\} \in \Theta_w^k$ ne.

Let $w \notin s$. Then for some $w' \in s$ we have $w' \in s_w$. Since $s_w \in \Theta_{s-w}^k$ ne, we have $s_w \in \Theta_{s-w}^k$.
 So $w \in \Theta_{s-w}^k$ whence $w \in \Theta_s^k$.

So $w \in s \iff \{w\} \in \Theta_{s-w}^k$ ne.
 And $\{w\} \in \Theta_{s-w}^k \iff w \in s$ ne.
 Therefore $s \in \Theta_s^k \iff \left(\binom{k}{w} \text{ ne} \right)$, i.e. $s \in \Theta_s^k$.

Let $w \notin s$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$$

$$s \in \Theta_s^k \iff s - k \leq s$$

Proof.

Case 1 : $s = 0$.
Then $\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right)_{w \leq 0}$ and $s - k \leq s$ iff $s \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.
Want to show: $s \in \Theta_s^k$ implies there are $s_w \leq s$ ($w \leq s$) s.t. $s_w \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$ and $s = \sum_{w \leq s} s_w$.

Case 2: $s > 0$.
Want to show:
 $\sum_{w \leq s} s_w = s$ and $\sum_{w \leq s} s_w - k \leq \sum_{w \leq s} s_w$.

Let $w \leq s$. Let $w \leq s$ be s.t. $w - k \leq w$.
Then $w \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$ and $\{w\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.

Let $w \leq s$. Then for some $w \leq s$ we have $w \leq s_w$. Since $s_w \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$, we have $s_w \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$. So $w \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$ whence $w - k \leq w$.

So $w \leq s$ $\{w\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.
And $\{w\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$ $w \leq s$ $\{w\} \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.
Therefore $s \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$, i.e. $s \in \Theta_s^k$.

Let $w \leq s$. Then there is some $s_w \leq s$ such that $s_w \in \left(\binom{k}{w} \text{ ne} \right)_{w \leq s}$.

$$\Theta_s^k = \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)_{w \ s}$$

$$s \left(\Theta_s^k \right) \quad s - k \ S$$

Proof.

Case 1 : $s = \cdot$

Then $\Theta_s^k =$ and

$$s - k \ S \quad s = \quad s \left(\cdot \right)$$

:

$s \left(\Theta_s^k \right)$ implies there are $s_w \ s \left(\begin{matrix} k \\ w \end{matrix} \ S \right)$ s.t.
 $s_w \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$ and $s = \quad w \ s \ S_w$.

Case 2: $s \quad \cdot$

:

Let $w \ s$. Let $w \ s$ be s.t. $w - k \ w$.

Then $w \left(\begin{matrix} k \\ w \end{matrix} \right)$
 so $\{w\} \left(\begin{matrix} k \\ w \end{matrix} \right)$ and $\{w\} \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$.

So $w \ s \ \{w\} \ s \ \{w\} \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$.
 And $\{w\} \ s \ w \ s \ \{w\} \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$.
 Therefore $s \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$, i.e. $s \left(\Theta_s^k \right)$.

Want to show:

$$\begin{matrix} W \ S & W \ S & W - k \ W \\ W \ S & W \ S & W - k \ W \end{matrix}$$

Let $w \ s$. Then for some $w \ s$ we have
 $w \ s_w$. Since $s_w \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$, we have $s_w \left(\begin{matrix} k \\ w \end{matrix} \right)$.
 So $w \left(\begin{matrix} k \\ w \end{matrix} \right)$ whence $w - w$.

Let $w \ s$. Then there is some $s_w \ s$ such that
 $s_w \left(\begin{matrix} k \\ w \end{matrix} \text{ ne} \right)$. Since $s_w \left(\text{ne} \right)$, there is some
 $w \ s_w$.

$$\Theta_s^k = \left(\binom{k}{w} \text{ ne} \right) \quad s \left(\Theta_s^k \quad s - k S \right)$$

Proof.

Case 1 : $s = \dots$
 Then $\Theta_s^k = \dots$ and $s - k S = \dots$
 $s = \dots$ $s \left(\dots \right)$
 :
 Let $W \in s$. Let $W \in s$ be s.t. $W - k W$.
 Then $W \left(\binom{k}{w} \right)$ so $\{W\} \left(\binom{k}{w} \right)$ and $\{W\} \left(\binom{k}{w} \text{ ne} \right)$.
 So $W \in s \implies \{W\} \in s \implies \{W\} \left(\binom{k}{w} \text{ ne} \right)$.
 And $\{W\} \in s \implies W \in s \implies \{W\} \left(\binom{k}{w} \text{ ne} \right)$.
 Therefore $s \left(\left(\binom{k}{w} \text{ ne} \right) \right)$, i.e. $s \left(\Theta_s^k \right)$.

Want to show:
 $W \in s \implies W - k W$
 $W \in s \implies W \in s \implies W - k W$.
 □

Let $W \in s$. Then for some $W \in s$ we have $W \in s_w$. Since $s_w \left(\binom{k}{w} \text{ ne} \right)$, we have $s_w \left(\binom{k}{w} \right)$. So $W \left(\binom{k}{w} \right)$ whence $W - k W$.

Let $W \in s$. Then there is some $s_w \in s$ such that $s_w \left(\binom{k}{w} \text{ ne} \right)$. Since $s_w \in s$, there is some $W \in s_w$. Then $W \left(\binom{k}{w} \right)$ and so $W - k W$.

Characteristic formulas for **properties**
(disjunctive normal form):

for P invariant under k -bisimulation:

$$M ; s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right) \quad (M ; s) P$$

Characteristic formulas for **properties**
(disjunctive normal form):

for P invariant under k -bisimulation:

$$M ; s \left(\bigvee_{(M;s) \models P} \Theta_s^k \right) \quad (M ; s) \models P$$

Theorem

$$\{ \text{BSML} \} = \{ \text{property } P \mid P \text{ is invariant under team } k\text{-bisimulation for some } k \in \mathbb{N} \}$$

Property P is *union closed*:

$$\{(M; s_i) \mid i \in I\} \models P \quad \Leftrightarrow \quad (M; \bigcup_{i \in I} s_i) \models P$$

Property P is *union closed*:

$$\{(M; s_i) \mid i \in I\} \models P \quad \text{implies} \quad (M; \bigcup_{i \in I} s_i) \models P$$

Property P has the *empty team property*:

$$(M; s) \models P \quad \text{implies} \quad (M; \text{)} \models P$$

Property P is *union closed*:

$$\{(M; s_i) \mid i \in I\} \models P \quad \text{iff} \quad (M; \bigcup_{i \in I} s_i) \models P$$

Property P has the *empty team property*:

$$(M; s) \models P \quad \text{iff} \quad (M; \emptyset) \models P$$

Theorem

$$\begin{aligned} \mathcal{U} &= \{ P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N} \} \\ &= \text{BSML} \end{aligned}$$

$BSML$ is union closed, but not expressively complete for

$U = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$.

$BSML$ is union closed, but not expressively complete for

$U = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}.$

Lemma

For $BSML$: has the empty team property is downward closed.

$BSML$ is union closed, but not expressively complete for

$\mathcal{U} = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$.

Lemma

For $BSML$: \mathcal{U} has the empty team property \mathcal{U} is downward closed.

Consider $(p \text{ ne}) \cup (\neg p \text{ ne}) \in \mathcal{U}$.

Assume $(p \text{ ne}) \cup (\neg p \text{ ne}) \notin \mathcal{U}$ for $BSML$.

$BSML$ is union closed, but not expressively complete for

$U = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}.$

Lemma

For $BSML$: $(p \vee ne) \wedge (\neg p \vee ne)$ has the empty team property and \bigcup is downward closed.

Consider $(p \vee ne) \wedge (\neg p \vee ne) \in U$.

Assume $(p \vee ne) \wedge (\neg p \vee ne) \notin BSML$.

If $\{w_p; w_{\neg p}\} \models (p \vee ne) \wedge (\neg p \vee ne)$, then $\{w_p; w_{\neg p}\} \models \bigcup$.

$BSML$ is union closed, but not expressively complete for

$\cup = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}$.

Lemma

For $BSML$: \cup has the empty team property \cup is downward closed.

Consider $(p \text{ ne}) \cup (\neg p \text{ ne}) \in \cup$.

Assume $\cup = (p \text{ ne}) \cup (\neg p \text{ ne})$ for $BSML$.

If $\{w_p; w_{\neg p}\} \models (p \text{ ne}) \cup (\neg p \text{ ne})$, then $\{w_p; w_{\neg p}\} \models \cup$.

By downward closure $\{w_p\} \models \cup$.

$\{w_p\} \models (p \text{ ne}) \cup (\neg p \text{ ne})$, a contradiction.

$BSML$ is union closed, but not expressively complete for

$\cup = \{P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N}\}.$

Lemma

For $BSML$: \cup has the empty team property \cup is downward closed.

Consider $(p \text{ ne}) \cup (\neg p \text{ ne}) \cup$.

Assume $\cup = (p \text{ ne}) \cup (\neg p \text{ ne})$ for $BSML$.

If $\{w_p; w_{\neg p}\} \Vdash (p \text{ ne}) \cup (\neg p \text{ ne})$, then $\{w_p; w_{\neg p}\} \Vdash$.

By downward closure $\{w_p\} \Vdash$.

$\{w_p\} \Vdash (p \text{ ne}) \cup (\neg p \text{ ne})$, a contradiction.

In $BSML$: $(p \text{ ne}) \cup (\neg p \text{ ne}) = ((p \text{ ne}) \cup (\neg p \text{ ne}))$

$$S \left(\Theta_S^k \right)$$

$$S \left(\Theta_S^k \right)$$

$$S - k S$$

$$S - k S \text{ or } S =$$

$$s \models \Theta_s^k \quad S =_k S$$

$$s \models \Theta_s^k \quad S =_k S \text{ or } S =$$

Characteristic formulas for union-closed properties with the empty team property:

$$M ; s \models \Theta_s^k \quad (M ; s) \models P$$

Characteristic formulas for union-closed properties without the empty team property:

$$M ; s \models \left(\Theta_s^k \right) \text{ ne } (M ; s) \models P$$

$$s \models \Theta_s^k$$

$$s \sim_k s$$

$$s \models \Theta_s^k$$

$$s \sim_k s \text{ or } s = \epsilon$$

Characteristic formulas for **union-closed properties with the empty team property**:

$$M ; s \models \left(\bigvee_{(M;s) \models P} \Theta_s^k \right) \iff (M ; s) \models P$$

Characteristic formulas for **union-closed properties without the empty team property**:

$$M ; s \models \left(\bigvee_{(M;s) \models P} \Theta_s^k \right) \text{ ne } \iff (M ; s) \models P$$

Theorem

$$\{ \text{BSML} \}$$

=

$$\{ P \mid P \text{ is union closed and invariant under } k\text{-bisimulation for some } k \in \mathbb{N} \}$$

For P union-closed; with the empty team property; invariant under k -bisimulation:

$$M ; s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right) (M ; s) P$$

Proof.

: Let $(M ; s) P$.

For P union-closed; with the empty team property; invariant under k -bisimulation:

$$M ; s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right)$$

Proof.

: Let $(M ; s) P$.

$s \left(\Theta_{s'}^k \right)$ so $s \left(\Theta_{s'}^k \right)$.

For P union-closed; with the empty team property; invariant under k -bisimulation:

$$M ; s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right) (M ; s) P$$

Proof.

: Let $(M ; s) P$.

$s \left(\Theta_{s'}^k \right)$ so $s \left(\Theta_s^k \right)$.

$(M ; s) P \left(\Theta_s^k \right)$.

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{\text{oe}}, s^{\text{oe}} \left(\bigwedge_{s'} \left(M; s \cdot > P \implies M^{\text{oe}}, s^{\text{oe}} > P \right) \right)$$

\hat{O} : Let $\hat{M}; s \cdot > P$.

$s^{\text{oe}} \left(\bigwedge_{s'} \left(s \text{ so } s^{\text{oe}} \left(\bigwedge_{s'} \left(M; s \cdot > P \implies M^{\text{oe}}, s^{\text{oe}} > P \right) \right) \right) \right)$

$\bigwedge_{s'} \left(\hat{M}; s \cdot > P \implies g \left(\bigwedge_{s'} \left(M; s \cdot > P \implies M^{\text{oe}}, s^{\text{oe}} > P \right) \right) \right)$

$s^{\text{oe}} \text{ so } s^{\text{oe}} g$.

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{oe}; s^{oe} (\hat{M}; s^{\bullet} > P \wedge \frac{k}{s} \hat{M}; s^{oe} > P)$$

$\hat{O} : \text{Let } \hat{M}; s^{\bullet} > P.$
 $s^{oe} (\frac{k}{s^{oe}} s^{oe} s^{oe} (\wedge \frac{k}{s^{oe}} \hat{M}; s^{\bullet} > P \wedge \frac{k}{s} \hat{M}; s^{oe} > P))$
 $s^{oe} s^{oe} g (\wedge \frac{k}{s} \hat{M}; s^{\bullet} > P)$
 $s^{oe} s^{oe} g.$
 Therefore
 $s^{oe} (\hat{M}; s^{\bullet} > P \wedge \frac{k}{s} \hat{M}; s^{oe} > P)$

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{oe}; s^{oe} \left(\bigwedge_{s \in M} \bigwedge_{s \bullet} \bigwedge_{s \bullet} M^{oe}; s^{oe} \right) > P$$

$$\hat{O} : \text{Assumes}^{oe} \left(\bigwedge_{M; s \bullet > P} \bigwedge_{s \bullet} \bigwedge_{s \bullet} \right)$$

$\hat{O} : \text{Let } \bigwedge_{M; s \bullet > P}.$

$$s^{oe} \left(\bigwedge_{s \bullet} \bigwedge_{s \bullet} s^{oe} \right) \wedge \bigwedge_{s \bullet} \bigwedge_{s \bullet}$$

$$! \bigwedge_{M; s \bullet > P} g \left(\bigwedge_{s \bullet} \bigwedge_{s \bullet} \right)$$

$$s^{oe} \bigwedge_{s \bullet} g.$$

Therefore

$$s^{oe} \left(\bigwedge_{M; s \bullet > P} \bigwedge_{s \bullet} \bigwedge_{s \bullet} \right)$$

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{\text{oe}}, s^{\text{oe}} \left(\bigwedge_{s'} \left(M; s \cdot > P \right) \wedge \frac{k}{s} \right) \wedge M^{\text{oe}}, s^{\text{oe}} > P$$

\hat{O} : Let $\hat{M}^{\text{oe}}, s^{\text{oe}} > P$.

$$s^{\text{oe}} \left(\frac{k}{s^{\text{oe}}} \text{ so } s^{\text{oe}} \left(\bigwedge_{s'} \frac{k}{s'} \right) \right)$$

$$! \hat{M}; s \cdot > P \text{ g } \left(\bigwedge_{s'} \frac{k}{s'} \right)$$

$$s^{\text{oe}} \text{ s }^{\text{oe}} \text{ g}.$$

Therefore

$$s^{\text{oe}} \left(\hat{M}; s \cdot > P \wedge \frac{k}{s} \right)$$

$$\hat{O} : \text{Assumes}^{\text{oe}} \left(M; s > P \wedge \frac{k}{s} \right)$$

Then $s^{\text{oe}} \text{ T}$ where

$$! t > T \text{ t } \text{ g } \text{ or } \hat{M}_t; s_t \cdot > P \text{ t } \left(\frac{k}{s_t} \right)$$

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{\text{oe}}, s^{\text{oe}} \left(\bigwedge_{s'} \hat{M}; s \cdot > P \right)$$

\hat{O} : Let $\hat{M}; s^{\text{oe}} > P$.

$$s^{\text{oe}} \left(\bigwedge_{s'} s \text{ so } s^{\text{oe}} \left(\bigwedge_{s'} \right) \right)$$

$$! \hat{M}; s \cdot > P \text{ g } \left(\bigwedge_{s'} \right)$$

$$s^{\text{oe}} \text{ s }^{\text{oe}} \text{ g}.$$

Therefore

$$s^{\text{oe}} \left(\bigwedge_{s'} \hat{M}; s \cdot > P \right) \wedge \bigwedge_{s'} \left(\right)$$

$$\hat{O} : \text{Assumes}^{\text{oe}} \left(\bigwedge_{s'} \hat{M}; s \cdot > P \right)$$

Then $s^{\text{oe}} \text{ T}$ where

$$! t > T \text{ t } \text{ g or } \hat{M}_t; s_t \cdot > P \text{ t } \left(\bigwedge_{s'} \right)$$

$$! t > T \text{ t } \text{ g or } \hat{M}_t; s_t \cdot > P \text{ t } - \bigwedge_{s'} \left(\right)$$

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{oe}, s^{oe} \left(\bigwedge_{s'} \hat{M}; s' \triangleright P \right) \wedge \frac{k}{s} \hat{M}; s^{oe} \triangleright P$$

\hat{O} : Let $\hat{M}; s^{oe} \triangleright P$.

$$s^{oe} \left(\frac{k}{s^{oe}} s o s^{oe} \left(\bigwedge_{s'} \frac{k}{s'} \right) \right)$$

$$! \hat{M}; s' \triangleright P \quad g \left(\bigwedge_{s'} \frac{k}{s'} \right)$$

$$s^{oe} s^{oe} g.$$

Therefore

$$s^{oe} \left(\bigwedge_{s'} \hat{M}; s' \triangleright P \right) \wedge \frac{k}{s} \hat{M}; s^{oe} \triangleright P.$$

$$\hat{O} : \text{Assumes}^{oe} \left(\bigwedge_{s'} \hat{M}; s' \triangleright P \right) \wedge \frac{k}{s} \hat{M}; s^{oe} \triangleright P.$$

Then $s^{oe} \triangleright T$ where

$$\begin{aligned} ! t \triangleright T \quad t \quad g \text{ or } \xi \hat{M}_t; s_t \triangleright P \quad t \left(\frac{k}{s_t} \right) \\ ! t \triangleright T \quad t \quad g \text{ or } \xi \hat{M}_t; s_t \triangleright P \quad t - k \quad s_t \end{aligned}$$

If $! t \triangleright T \quad t \quad g$, then $s^{oe} \triangleright g \triangleright P$ by the empty team property.

For P union-closed; with the empty team property; invariant under bisimulation:

$$M^{oe}; s^{oe} \left(\bigwedge_{s \in M} s \right) \wedge \bigwedge_{s \in M} s^{oe} > P$$

Proof

\hat{O} : Assumes^{oe}($M; s > P \wedge \bigwedge_{s \in M} s$.

\hat{O} : Let $\hat{M}; s^{oe} > P$.

$s^{oe} \left(\bigwedge_{s \in M} s \right) \wedge \bigwedge_{s \in M} s^{oe}$

$\vdash \hat{M}; s > P \text{ g } \left(\bigwedge_{s \in M} s \right)$

$s^{oe} \text{ s } \text{ g}$.

Therefore $s^{oe} \left(\hat{M}; s > P \wedge \bigwedge_{s \in M} s \right)$

Then $s^{oe} \text{ T}$ where

$\vdash t > T \text{ t } \text{ g or } \hat{M}_t; s_t > P \text{ t } \left(\bigwedge_{s \in M} s \right)$

$\vdash t > T \text{ t } \text{ g or } \hat{M}_t; s_t > P \text{ t } - \bigwedge_{s \in M} s$

If $\vdash t \text{ t } \text{ g}$, then $s^{oe} \text{ g } > P$ by the empty team property.

Otherwise let $T^{oe} \sim t > T \text{ St } \times \text{ g}$ and consider $M \ \& \ \sim M_t \ \text{ St } > T^{oe}$ and its team $\sim s_t \ \text{ St } > T^{oe}$.

For P union-closed; with the empty team property; invariant under k -bisimulation:

$$M ; s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right)$$

Proof.

: Assume $s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right)$.

: Let $(M ; s) P$.

$$s \left(\Theta_{s'}^k \text{ so } s \left(\Theta_{s'}^k \right)$$

$$(M ; s) P \left(\Theta_s^k \right)$$

$$s = s$$

Therefore

$$s \left(\begin{array}{c} \Theta_s^k \\ (M ; s) P \end{array} \right)$$

Then $s = T$ where

$$\begin{array}{l} t \in T \text{ } t = \text{ or } (M_t ; s_t) P \text{ } t \left(\Theta_{s_t}^k \right) \\ t \in T \text{ } t = \text{ or } (M_t ; s_t) P \text{ } t \text{ - }_k \text{ } s_t \end{array}$$

If $t \in T$, then $s = P$ by the empty team property. □

Otherwise let $T = \{t \in T \mid t \in T\}$ and consider $M = \{M_t \mid t \in T\}$ and its team $u = \{s_t \mid t \in T\}$. $(M ; u) P$ by invariance and union closure.

And $s \text{ - }_k u$ so $s \in P$.

$\sim_{SSSS} \S \triangleright BSMLI \bullet$

$\sim_{P \ SP} \text{ is invariant under } k\text{-bisimulation for some } k > N \bullet$

Characteristic formulas: $\bigwedge_{s \leq k} M; s \triangleright P$

$\sim_{SSSS} \S \triangleright BSMLE \bullet$

$\sim_{P \ SP} \text{ is union closed and invariant under } k\text{-bisimulation for some } k > N \bullet$

Characteristic formulas: $\bigwedge_{s \leq k} M; s \triangleright P$ and $\bigwedge_{s \leq k} M; s \triangleright P \bullet, ne$

BSML¹ axiomatization

and : classical formulas (none or 1).

Non-modal portion (adapted from the system for DT):

introduction

$$\frac{D^\ddagger}{\text{---}} \text{I}^\ddagger$$

elimination

$$\frac{D_1 \quad D_2}{\text{---}} \text{E}$$

I^\ddagger The undischarged assumptions in D^\ddagger do not contain ne.

, introduction

$$\frac{D_1 \quad D_2}{\quad}, I$$

, elimination

$$\frac{D}{\quad}', E$$

$$\frac{D}{\quad}', E$$

1 introduction

$$\frac{D}{1} 1 I$$

$$\frac{D}{1} 1 I$$

1 elimination

$$\frac{D \quad D_1 \quad D_2}{1} 1 E$$

- weak introduction

$$\frac{D}{-} - I^{\wedge \ddagger \bullet}$$

- weakening

$$\frac{D}{-} - W$$

- weak elimination

$$\frac{D \quad D_1^{\ddagger} \quad D_2^{\ddagger}}{-} - E^{\wedge \ddagger ; \bullet}$$

- weak substitution

$$\frac{D \quad D_1^{\ddagger}}{-} - Sub^{\wedge \ddagger \bullet}$$

$\wedge \ddagger \bullet$ The undischarged assumptions $iD^{\ddagger}; D_2^{\ddagger}$ do not contain ne.
 $\wedge \ddagger \bullet$ may not contain ne.
 $\wedge ; \bullet$ may not contain 1 outside the scope of an .

- commutativity

$$\frac{D}{-} \text{Com}$$

- associativity

$$\frac{D}{- \quad \wedge \quad - \quad \bullet \quad -} \text{Ass}$$

-1 distributivity

$$\frac{D}{\wedge \quad - \quad \bullet \quad 1 \quad \wedge \quad - \quad \bullet} \text{Distr - 1}$$

– Elimination

D

--

———— –E

á –, ne

á elimination

D
á
—— á E

á contraction

D
á -
———— á Ctr

ne introduction

$$\frac{}{-1 \text{ ne}} \text{ nel}$$

- ne elimination

	D		D ₁		D ₂		$\hat{}, \text{ne}^\bullet - \hat{}, \text{ne}^\bullet$		D ₃	
-										- neE

New rules for :

<p>ne elimination</p> $\frac{\frac{D}{\text{ne}}}{-} \text{neE}$	<p>Double elimination</p> $\frac{D}{\text{DN}}$	
<p>De Morgan 1</p> $\frac{\frac{D}{\hat{\ , \bullet}}}{-} \text{DM}_1$	<p>De Morgan 2</p> $\frac{\frac{D}{\hat{\ - \bullet}}}{\ ,} \text{DM}_2$	<p>De Morgan 3</p> $\frac{\frac{D}{\hat{\ 1 \bullet}}}{\ ,} \text{DM}_3$

Modal portion|basic rules:

n monotonicity

$$\frac{D^{\text{oe}} \quad D}{n \quad n} \quad n \text{ Mon}^{\wedge} \dagger \bullet$$

j monotonicity

$$\frac{D^{\text{oe}} \quad D_1 \quad \dots \quad D_n}{j_1 \quad \dots \quad j_n \quad j} \quad j \text{ Mon}^{\wedge} \dagger \bullet$$

nj interaction

$$\frac{D}{n \quad j} \quad \text{Inter } nj$$

[^]†• D^{oe} does not contain undischarged assumptions.

New modal rules:

n 1- conversion

$$\frac{D \quad n^{\wedge} 1 \bullet}{n \quad -n} \text{Conv}n \ 1-$$

j 1- conversion

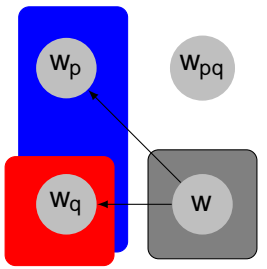
$$\frac{D \quad j^{\wedge} 1 \bullet}{j \quad -j} \text{Conv}j \ 1-$$

<p>n separation</p> $\frac{n^{\wedge} - \hat{\wedge}, ne^{\bullet\bullet}}{n} \text{ n Sep}$	<p>n join</p> $\frac{D_1 \quad D_2}{n^{\wedge} - \bullet} \text{ n Join}$
<p>j instantiation</p> $\frac{j^{\wedge}, ne^{\bullet}}{n} \text{ j Inst}$	<p>jn join</p> $\frac{D_1 \quad D_2}{j^{\wedge} - \bullet} \text{ jn Join}$

s (n | w > s \$t b R w t x g and t (

s (j | w > s R w (

$$\frac{n^{\hat{p}} - n^{\hat{q}}, ne^{\bullet\bullet}}{n} \text{ n Sep}$$

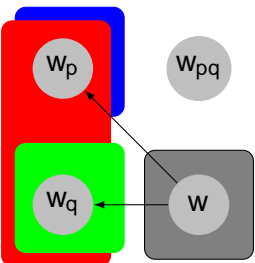
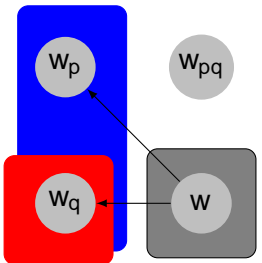


$s(n^{\hat{p}} - n^{\hat{q}}, ne^{\bullet\bullet})$
 $s(n^{\hat{q}})$

$s(n \quad | w > s \quad \&t b R w \quad t x g \text{ and } t ($
 $s(j \quad | w > s \quad R w ($

$$\frac{n^{\wedge} - \hat{\wedge}, ne^{\bullet\bullet}}{n} \text{ n Sep}$$

$$\frac{D_1 \quad D_2}{n \quad n} \text{ n Join}$$



s (n^{\wedge} p - \hat{\wedge} q, ne^{\bullet\bullet})
s (n q)

s (n p, n q)
s (n^{\wedge} p - q^{\bullet})

s (n \quad | w > s \quad \&t b R w \quad t x g \text{ and } t ()
s (j \quad | w > s \quad R w ()

oooooooooooooooooooooooo

oooooooooooo

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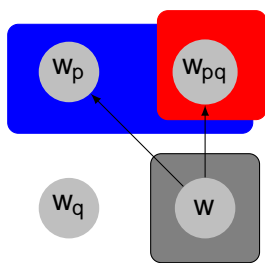
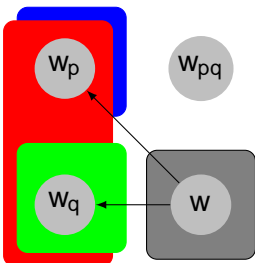
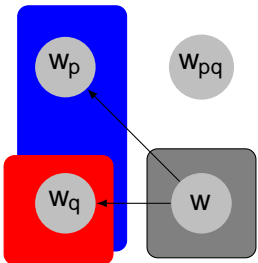
oooo

oooooooo

$$\frac{n^{\wedge} - \hat{\wedge}, ne^{\bullet\bullet}}{n} \text{ n Sep}$$

$$\frac{D_1 \quad D_2}{n^{\wedge} - \bullet} \text{ n Join}$$

$$\frac{D_1 \quad D_2}{j^{\wedge} - \bullet} \text{ j n Join}$$



s (n^{\wedge} p - \hat{\wedge} q, ne^{\bullet\bullet})
s (n q)

s (n p, n q)
s (n^{\wedge} p - q^{\bullet})

s (j p, n q)
s (j^{\wedge} p - q^{\bullet})

s (n | w > s § t b R w t x g and t ()
s (j | w > s R w ()

$n \hat{\hat{}}$, $ne^\bullet - \hat{\hat{}}$, $ne^{\bullet\bullet}$ $\cup \emptyset$ n , n FC

$$\frac{D}{n} \quad \frac{j \hat{\hat{}} , ne^\bullet}{j \text{ Inst}}$$

Corresponds to "j • (n
 \ought implies may" for pragmatically enriched formulas.

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\forall \varphi \in \text{BSML}^1 \exists \hat{O} \text{ s.t. } \hat{O} \models \varphi \text{ iff } \varphi \text{ is satisfiable in } \mathcal{M}^k$

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\exists \text{BSML}^1 \hat{O} \vdash k \text{ Cmodal depth} \hat{h} \bullet \wp P \quad \bigcup \emptyset \quad \begin{matrix} k \\ s \end{matrix}$
 $\hat{M}; s \bullet \succ P$

(

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\exists \text{BSML}^1 \hat{O} \vdash_k \text{Cmodal depth} \hat{h} \bullet \exists P \quad \bigcup_{\hat{M}; s \succ P} \hat{O}^k$

$(\hat{O}^k)_{\hat{M}; s \succ P} (\hat{N}; t \succ Q)^k$

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\exists \text{BSML}^1 \hat{O} \vdash k \text{ Cmodal depth} \bullet \exists P \bigcup_{\hat{M};s \succ P} \overset{k}{s}$

$$\left(\hat{O} \bigcup_{\hat{M};s \succ P} \overset{k}{s} \right) \left(\hat{N};t \succ Q \overset{k}{t} \right)$$

$$\hat{O} \vdash \hat{M};s \succ P \hat{\exists} \hat{N};t \succ Q \quad s - k t$$

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\hat{O} \models \text{BSML}^1 \iff \exists k \text{ Cmodal depth } \hat{O} \bullet \exists P \exists \emptyset \begin{matrix} k \\ s \end{matrix} \hat{M}; s \bullet \triangleright P$

$(\hat{O} \models \text{BSML}^1) \iff \exists k \left(\begin{matrix} k \\ s \end{matrix} \hat{M}; s \bullet \triangleright P \wedge \begin{matrix} k \\ t \end{matrix} \hat{N}; t \bullet \triangleright Q \right)$

$\hat{O} \models \text{BSML}^1 \iff \exists k \left(\begin{matrix} k \\ s \end{matrix} \hat{M}; s \bullet \triangleright P \wedge \begin{matrix} k \\ t \end{matrix} \hat{N}; t \bullet \triangleright Q \wedge \begin{matrix} k \\ s \end{matrix} \hat{O} \models \begin{matrix} k \\ t \end{matrix} \right)$

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\hat{O} \models \Box^k P \iff \hat{O} \models \bigvee_{s \in S} \Box^k P_s$

$$\hat{O} \models \Box^k P \iff \hat{O} \models \bigvee_{s \in S} \Box^k P_s$$

$$\hat{O} \models \Box^k P \iff \hat{O} \models \bigvee_{s \in S} \Box^k P_s$$

$$\hat{O} \models \Box^k P \iff \hat{O} \models \bigvee_{s \in S} \Box^k P_s$$

Completeness

Use the disjunctive normal form and a strategy developed in [7]:

Lemma: $\hat{O} \models \bigvee_{k \in \mathbb{N}} \text{Cmodal depth } k \cdot \hat{P} \iff \bigcup_{s \in S} \hat{M}; s \triangleright P$

$$\hat{O} \models \bigvee_{s \in S} \hat{M}; s \triangleright P \iff \bigvee_{t \in T} \hat{N}; t \triangleright Q$$

$$\hat{O} \models \hat{M}; s \triangleright P \iff \hat{N}; t \triangleright Q \iff \bigcup_{s \in S} \bigvee_{t \in T} \hat{O} \models \hat{M}; s \triangleright P \iff \bigcup_{t \in T} \hat{O} \models \hat{N}; t \triangleright Q$$

$$\hat{O} \models \hat{M}; s \triangleright P \iff \hat{N}; t \triangleright Q \iff \hat{O} \models \hat{M}; s \triangleright P \iff \hat{O} \models \hat{N}; t \triangleright Q$$

BSML[^] axiomatization

Exclude 1 -rules and neE; and add:

BSML[^] axiomatization

Exclude 1- rules and neE; and add:

^ 1-

BSML[^]

BSML¹

^ introduction

$\frac{D}{\wedge} \wedge I$	$\frac{D}{\wedge} \wedge I$
-----------------------------	-----------------------------

1 introduction

$\frac{D}{1} 1 I$	$\frac{D}{1} 1 I$
-------------------	-------------------

BSML[^] axiomatization

Exclude 1- rules and neE; and add:

\wedge 1 -

BSML[^]

BSML¹

\wedge introduction

$$\frac{D}{\wedge} \wedge I \qquad \frac{D}{\wedge} \wedge I$$

\wedge ne introduction

$$\frac{}{\wedge ne} \wedge neI$$

1 introduction

$$\frac{D}{1} 1I \qquad \frac{D}{1} 1I$$

ne introduction

$$\frac{}{-1 ne} neI$$

BSML[^] axiomatization

Exclude 1- rules and- neE; and add:

^ 1- ^

BSML[^]

<p>^ introduction</p> $\frac{D}{\wedge} \wedge I$ <p>^ ne introduction</p> $\frac{\wedge ne}{\wedge} \wedge ne I$	<p>D</p> $\frac{D}{\wedge} \wedge I$ <p>^ introduction</p> $\frac{D}{\wedge} \wedge I$
---	--

BSML¹

<p>1 introduction</p> $\frac{D}{1} 1 I$ <p>ne introduction</p> $\frac{-1 ne}{-1} ne I$	<p>D</p> $\frac{D}{1} 1 I$
--	----------------------------

BSML[^][^] elimination

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \text{D} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \sim \text{---} \text{ ; } m \bullet \\ \text{D}_1 \end{array} & \begin{array}{c} \text{---} \sim \text{---} \text{ ; } m \bullet \\ \text{D}_2 \end{array} \\
 \end{array} \\
 \hline
 \text{---} \text{E} \text{---} \ddagger \bullet
 \end{array}$$

$\ddagger \bullet$ The occurrence at index m is not within the scope of n .

BSML¹

1 elimination

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \text{D} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{D}_1 \end{array} & \begin{array}{c} \text{---} \\ \text{D}_2 \end{array} \\
 \end{array} \\
 \hline
 \text{---} \text{---} \text{---} \text{E}
 \end{array}$$

BSML[^]

BSML¹

n[^] elimination

$$\frac{
 \begin{array}{c}
 D \\
 n
 \end{array}
 \quad
 \begin{array}{c}
 \sim \sim \wedge ; m \bullet \\
 D_1 \\
 1
 \end{array}
 \quad
 \begin{array}{c}
 \sim \sim \wedge ; m \bullet \\
 D_2 \\
 2
 \end{array}
 }{
 \begin{array}{c}
 n \quad 1 - n \quad 2 \\
 n \wedge E^{\ddagger} \bullet
 \end{array}
 }$$

$\wedge^{\ddagger} \bullet$ The occurrence at index m is not within the scope of a modality which occurs in \sim , and not within the scope of \wedge (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

n¹- conversion

$$\frac{
 \begin{array}{c}
 D \\
 n \wedge 1 \bullet \\
 n - n
 \end{array}
 }{
 \text{Conv}n 1-
 }$$

BSML[^]

BSML¹

n[^] elimination

$$\frac{\begin{array}{c} D \\ n \end{array} \quad \begin{array}{c} \sim \sim \wedge ; m \bullet \\ D_1 \\ 1 \end{array} \quad \begin{array}{c} \sim \sim \wedge ; m \bullet \\ D_2 \\ 2 \end{array}}{n \quad 1 - n \quad 2} \quad n \wedge E \ddagger \bullet$$

j[^] elimination

$$\frac{\begin{array}{c} D \\ j \end{array} \quad \begin{array}{c} \sim \sim \wedge ; m \bullet \\ D_1 \\ 1 \end{array} \quad \begin{array}{c} \sim \sim \wedge ; m \bullet \\ D_2 \\ 2 \end{array}}{j \quad 1 - j \quad 2} \quad j \wedge E \ddagger \bullet$$

[^]‡• The occurrence at index *m* is not within the scope of a modality which occurs in *D*, and not within the scope of *j* (except if the forms part of *j*).

*D*₁; *D*₂ do not contain undischarged assumptions.

n¹- conversion

$$\frac{\begin{array}{c} D \\ n \wedge 1 \bullet \\ n \quad -n \end{array}}{\text{Conv}n \ 1-}$$

j¹- conversion

$$\frac{\begin{array}{c} D \\ j \wedge 1 \bullet \\ j \quad -j \end{array}}{\text{Conv}j \ 1-}$$

Completeness

Lemma:

$\exists \text{BSMLE } \hat{O} \quad \exists k \text{ Cmd}^k \cdot \exists P \quad \exists \emptyset \quad \wedge \quad \exists s \quad \text{or} \quad \exists \emptyset^k \quad \wedge \quad \exists s \cdot \text{, ne}$

$\wedge \quad \exists M; s \rightarrow P \quad \wedge \quad \exists M; s \rightarrow P$

(

Completeness

Lemma:

$\exists \text{BSMLE } \hat{O} \quad \exists \text{ } k \text{ Cmd}^{\wedge} \bullet \text{ } \xi P \quad \exists \emptyset \quad \wedge \quad k_s \quad \text{or} \quad \exists \emptyset^{\wedge} \quad \wedge \quad k_s \bullet, \text{ ne}$
$\begin{matrix} & \wedge \quad k_s \\ \hat{M};s \rightarrow P & \end{matrix}$
$\begin{matrix} \wedge \quad k_t \\ \hat{N};t \rightarrow Q \end{matrix}$

Completeness

Lemma:

$\hat{\Omega} \models \text{BSML} \iff \exists k \in \mathbb{N} \exists P \exists Q \left(\hat{\Omega} \models \bigwedge_{s \in S} \left(\bigwedge_{M; s \rightarrow P} \bigwedge_{N; t \rightarrow Q} \bigwedge_{s \in S} \left(\bigwedge_{t \in T} \dots \right) \right) \right)$

$\hat{\Omega} \models \bigwedge_{s \in S} \left(\bigwedge_{M; s \rightarrow P} \bigwedge_{N; t \rightarrow Q} \dots \right)$

Completeness

Lemma:

$$\begin{aligned} & \text{If } \hat{\Omega} \vdash_k \text{Cmd}^{\wedge} \bullet \text{ } \S P \text{ then } \hat{\Omega} \vdash_k \text{ } \wedge \text{ } \text{or } \hat{\Omega} \vdash_k \bullet, \text{ ne} \\ & \text{where } \hat{\Omega} \vdash_k \text{ } \wedge \text{ } \text{ means } \hat{M}; s \bullet \text{ } \wedge \text{ } \hat{N}; t \bullet \text{ } \wedge \text{ } \hat{M}; s \bullet \text{ } \wedge \text{ } \hat{N}; t \bullet \text{ } \wedge \text{ } \end{aligned}$$

$$\hat{\Omega} \vdash_k \hat{M}; s \bullet \text{ } \wedge \text{ } \text{R b Q} \text{ then } \hat{\Omega} \vdash_k \text{ } \wedge \text{ } \text{R}$$

$$\hat{N}; t \bullet \text{ } \wedge \text{ } \text{R b Q} \text{ then } \hat{N}; t \bullet \text{ } \wedge \text{ } \text{R}$$

Completeness

Lemma:

$$\Box \text{BSMLE } \hat{\Omega} \vdash_k \text{Cmd}^{\wedge} \bullet \wp P \quad \bigvee_{\hat{M};s \rightarrow P} \bigwedge_{k_s} \text{ or } \bigvee_{\hat{M};s \rightarrow P} \bigwedge_{k_s} \bullet, \text{ ne}$$

$$\left(\hat{\Omega} \wedge_{\hat{M};s \rightarrow P} \left(\bigwedge_{k_s} \left(\bigwedge_{k_t} \hat{N};t \rightarrow Q \right) \right) \right)$$

$$\hat{\Omega} \vdash \hat{M};s \rightarrow P \wp R \text{ b } Q \quad s - k \& R \quad \bigvee_{\hat{N};t \rightarrow R} \bigwedge_{k_t}$$

$$\bigvee_{\hat{N};t \rightarrow Q} \bigwedge_{k_t} \text{ Rb } Q \quad k \& R$$

Completeness

Lemma:

$$\hat{\Sigma} \vdash_{k, S} \text{Cmd}^k \bullet \text{P} \quad \text{or} \quad \hat{\Sigma} \vdash_{k, S} \text{ne}$$

$$\left(\hat{\Sigma} \vdash_{k, S} \text{P} \right) \wedge \left(\hat{\Sigma} \vdash_{k, T} \text{Q} \right)$$

$$\hat{\Sigma} \vdash_{k, S} \text{P} \quad \text{RbQ} \quad \text{S} - \text{k} \ \& \ \text{R}$$

$$\left(\hat{\Sigma} \vdash_{k, S} \text{Q} \right) \wedge \left(\hat{\Sigma} \vdash_{k, T} \text{R} \right)$$

$$\left(\hat{\Sigma} \vdash_{k, S} \text{Q} \right) \wedge \left(\hat{\Sigma} \vdash_{k, T} \text{Q} \right)$$

$$\left(\hat{\Sigma} \vdash_{k, T} \text{Q} \right) \wedge \left(\text{RbQ} \right) \wedge \text{R}$$

Completeness

Lemma:

$$\Box \text{BSMLE } \hat{\Omega} \quad ; k \text{ Cmd}^{\wedge} \bullet \xi P \quad \hat{U}\emptyset \quad \wedge \quad k_s \quad \text{or} \quad \hat{U}\emptyset^{\wedge} \quad \wedge \quad k_s \bullet, ne$$

$$\left(\hat{\Omega} \quad \wedge \quad k_s \quad \left(\quad \wedge \quad k_t \right. \right. \\ \left. \left. \quad \hat{M};s \bullet \rightarrow P \quad \hat{N};t \bullet \rightarrow Q \right) \right)$$

$$\hat{\Omega} \quad ; \hat{M};s \bullet \rightarrow P \quad \xi R \text{ b } Q \quad s - k \ \& \ R \\ k_s \emptyset \quad \wedge \quad k_t \\ \hat{N};t \bullet \rightarrow R \\ k_s \emptyset \quad \wedge \quad k_t \\ \hat{N};t \bullet \rightarrow Q \\ \wedge \quad k_s \emptyset \quad \wedge \quad k_t \\ \hat{N};t \bullet \rightarrow Q$$

$$\hat{N};t \bullet \rightarrow Q \quad \wedge \quad k_t \quad R \text{ b } Q \quad k \ \& \ R$$

Completeness

Lemma:

$$\text{BSMLE } \hat{\Omega} \vdash_k \text{Cmd}^{\wedge} \bullet \text{ } \xi P \quad \bigvee_{s \in S} \emptyset \quad \wedge \quad \text{or} \quad \bigvee_{s \in S} \emptyset^{\wedge} \quad \wedge \quad \text{ne}$$

$$\left(\hat{\Omega} \vdash_{M;s} \bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow Q \right) \right)$$

$$\hat{\Omega} \vdash_{M;s} \bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow Q \right) \text{ } \xi R \text{ } \& R$$

$$\bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow R \right) \text{ } \& R$$

$$\bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow Q \right) \text{ } \& R$$

$$\bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow Q \right) \text{ } \& R$$

$$\bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow Q \right) \text{ } \& R \text{ } \& R$$

$$\hat{\Omega} \vdash_{M;s} \bigwedge_{s \in S} \left(\bigwedge_{t \in T} \hat{N};t \rightarrow Q \right)$$

Completeness

Lemma:

$$\text{BSMLE } \hat{\Omega} \vdash_k \text{Cmd}^{\bullet} \cdot \xi P \quad \bigvee_{\hat{M};s \rightarrow P} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k \text{ or } \bigvee_{\hat{M};s \rightarrow P} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k, \text{ ne}$$

$$\left(\hat{\Omega} \vdash_{\hat{M};s \rightarrow P} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k \right) \left(\bigwedge_{\hat{N};t \rightarrow Q} \xi^k \right)$$

$$\hat{\Omega} \vdash_{\hat{M};s \rightarrow P} \xi R \text{ b } Q \quad \xi^k \text{ & } R$$

$$\bigvee_{\hat{N};t \rightarrow R} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k$$

$$\bigvee_{\hat{N};t \rightarrow Q} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k$$

$$\bigvee_{\hat{N};t \rightarrow Q} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k \quad R \text{ b } Q \quad \xi^k \text{ & } R$$

$$\hat{\Omega} \vdash_{\hat{M};s \rightarrow P} \bigwedge_{\hat{N};t \rightarrow Q} \xi^k \quad \bigwedge_{\hat{N};t \rightarrow Q} \xi^k \quad \hat{\Omega} \quad \emptyset$$

BSML axiomatization

Exclude \exists -rules and \neg e from BSML¹ and add:

BSML	BSML ¹
<p>\nege translation</p> $\frac{D \quad \frac{\Delta, \sim \phi ; m \bullet}{D_1} \quad \frac{\Delta, \neg \phi ; m \bullet}{D_2}}{\Delta, \neg \phi ; m \bullet} \neg e$ <p>ϕ The occurrence at index m is not within the scope of \exists or \neg.</p>	<p>\nege introduction</p> $\frac{\Delta, \phi ; m \bullet}{\Delta, \neg \phi ; m \bullet} \neg e$

BSML[^]

BSML¹

n- ne translation

n 1 - conversion

$$\frac{
 \begin{array}{c}
 D \\
 n
 \end{array}
 \quad
 \begin{array}{c}
 \hat{\cdot}, - \sim ; m \bullet \\
 D_1 \\
 1
 \end{array}
 \quad
 \begin{array}{c}
 \hat{\cdot}, ne \sim ; m \bullet \\
 D_2 \\
 2
 \end{array}
 }{
 \begin{array}{c}
 n - ne Trs \hat{\cdot} \ddagger \bullet \\
 n \quad 1 - n \quad 2
 \end{array}
 }$$

$$\frac{
 \begin{array}{c}
 D \\
 n \hat{\cdot} 1 \bullet \\
 n - n
 \end{array}
 }{
 \text{Conv n 1-}
 }$$

$\hat{\cdot} \ddagger \bullet$ The occurrence at index m is not within the scope of a modality which occurs in $\hat{\cdot}$, and not within the scope of \sim (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

BSML[^]

BSML¹

n- ne translation

n 1- conversion

$$\frac{\begin{array}{c} D \\ n \end{array} \quad \begin{array}{c} \hat{}, - \sim ; m \bullet \\ D_1 \\ 1 \end{array} \quad \begin{array}{c} \hat{}, ne \sim ; m \bullet \\ D_2 \\ 2 \end{array}}{n \quad 1 - n \quad 2} \quad n\text{- neTrs}^{\hat{}\bullet}$$

$$\frac{\begin{array}{c} D \\ n \end{array} \quad \begin{array}{c} \hat{} \\ 1 \end{array} \quad \bullet}{n \quad -n} \quad \text{Conv}n \ 1-$$

j- ne translation

j 1- conversion

$$\frac{\begin{array}{c} D \\ j \end{array} \quad \begin{array}{c} \hat{}, - \sim ; m \bullet \\ D_1 \\ 1 \end{array} \quad \begin{array}{c} \hat{}, ne \sim ; m \bullet \\ D_2 \\ 2 \end{array}}{j \quad 1 - j \quad 2} \quad j\text{- neTrs}^{\hat{}\bullet}$$

$$\frac{\begin{array}{c} D \\ j \end{array} \quad \begin{array}{c} \hat{} \\ 1 \end{array} \quad \bullet}{j \quad -j} \quad \text{Conv}j \ 1-$$

$\hat{}\bullet$ The occurrence at index m is not within the scope of a modality which occurs in $\hat{}$, and not within the scope of \sim (except if the forms part of j).

$D_1; D_2$ do not contain undischarged assumptions.

Old rules in [7]/[2] which are derivable:

k : set of all non-equivalent $\overset{k}{s_i}$ over \mathcal{S} , where $s_i \times g$
 ne

BSML

BSML¹

ne elimination

$$\begin{array}{c}
 \overset{k}{s_1} \sim ne; m \bullet \\
 D \quad D_1 \quad \dots \quad D_n \\
 \hline
 neE^{\hat{\ddagger}}
 \end{array}$$

$\hat{\ddagger}$ • The occurrence at index m is not within the scope of $\overset{k}{s_1}$ or \dots or $\overset{k}{s_n}$; $k > N$; $\sim s_1; \dots; s_n \bullet$.

1 elimination

$$\begin{array}{c}
 D \quad D_1 \quad D_2 \\
 1 \quad \quad \quad 1 E \\
 \hline
 1 E
 \end{array}$$

BSML

BSML¹

n ne elimination

$$\frac{\begin{array}{c} D \\ n \end{array} \quad \begin{array}{c} \wedge_{s_1}^k \sim ne; m \bullet \\ D_1 \\ 1 \end{array} \quad \dots \quad \begin{array}{c} \wedge_{s_n}^k \sim ne; m \bullet \\ D_n \\ n \end{array}}{i > n \quad i} \quad n \text{ ne } E^{\wedge \ddagger \bullet}$$

j ne elimination

$$\frac{\begin{array}{c} D \\ j \end{array} \quad \begin{array}{c} \wedge_{s_1}^k \sim ne; m \bullet \\ D_1 \\ 1 \end{array} \quad \dots \quad \begin{array}{c} \wedge_{s_n}^k \sim ne; m \bullet \\ D_n \\ n \end{array}}{i > j \quad i} \quad j \text{ ne } E^{\wedge \ddagger \bullet}$$

[^] ‡• The occurrence at index m is not within the scope of a modality which occurs in , and not within the scope of (except if the forms part of j); D₁;...; D_n do not contain undischarged assumptions; k > N; ~ s₁;...; s_n • k.

n 1- conversion

$$\frac{\begin{array}{c} D \\ n \end{array} \quad \begin{array}{c} \wedge_{s_1}^k \sim ne; m \bullet \\ 1 \\ -n \end{array}}{n \quad -n} \quad \text{Conv } n \text{ 1-}$$

j 1- conversion

$$\frac{\begin{array}{c} D \\ j \end{array} \quad \begin{array}{c} \wedge_{s_1}^k \sim ne; m \bullet \\ 1 \\ -j \end{array}}{j \quad -j} \quad \text{Conv } j \text{ 1-}$$



$\dagger \bullet$ The occurrence at index m is not within the scope of \sim or n ; $k > N$; $\sim_{s_1}; \dots; \sim_{s_n} \bullet^k$.

Let $\frac{k}{s}$ $\frac{k}{w}$. By classical completeness $\frac{k}{s}$, where s is such that for each k th Hintikka formula $\frac{k}{w}$ there is a $w > s$ such that $\frac{k}{w} \cup \emptyset \frac{k}{w}$



\ddagger The occurrence at index m is not within the scope of \sim or n ; $k > N$; $\sim_{s_1}; \dots; \sim_{s_n} \bullet^k$.

Let $\overset{k}{s} \supset_{w>s} \overset{k}{w}$. By classical completeness $\overset{k}{s}$, where s is such that for each k th Hintikka formula $\overset{k}{w}$ there is a $w > s$ such that $\overset{k}{w} \cup \emptyset \overset{k}{w}$.

Then it can be shown that $\emptyset \overset{k}{s}, ne \sim ne; m \bullet$. Let $\overset{k}{s}, ne \sim ne; m \bullet$.

$$\begin{array}{ccccccc}
 & & \hat{\Delta}_{s_1}^k \sim ne; m \bullet & & & & \hat{\Delta}_{s_n}^k \sim ne; m \bullet \\
 D & & D_1 & & \dots & & D_n \\
 \hline
 & & & & & & neE^{\dagger} \bullet
 \end{array}$$

$\hat{\dagger}$ • The occurrence at index m is not within the scope of Δ or n ; $k > N$; $\sim s_1, \dots, s_n \bullet$.

Let $\Delta_s^k \supseteq_{w>s} \Delta_w^k$. By classical completeness \mathcal{M}_s^k , where s is such that for each k th Hintikka formula Δ_w^k , there is a $w > s$ such that $\Delta_w^k \not\models \Delta_s^k$.

Then it can be shown that $\Delta_s^k \sim ne; m \bullet$. Let $\hat{\Delta}_s^k \sim ne; m \bullet$.

Consider the case in which $s = 2$. Let $\Delta_{w_1}^k - \Delta_{w_2}^k$.

$$\begin{array}{c}
 \overset{k}{s_1} \sim ne; m \bullet \\
 D_1 \\
 \vdots \\
 \overset{k}{s_n} \sim ne; m \bullet \\
 D_n \\
 \hline
 neE^{\ddagger} \bullet
 \end{array}$$

$\overset{k}{\ddagger} \bullet$ The occurrence at index m is not within the scope of \sim or n ; $k > N$; $\sim s_1; \dots; s_n \bullet$.

Let $\overset{k}{s}$ be a Hintikka formula. By classical completeness, there is such that for each $w > s$ there is a w such that $\overset{k}{w} \Vdash \overset{k}{s}$.

Then it can be shown that $\overset{k}{s}, ne \sim ne; m \bullet$. Let $\overset{k}{s}, ne \sim ne; m \bullet$.

Consider the case in which s is 2. Let $\overset{k}{s} = \overset{k}{w_1} - \overset{k}{w_2}$.

Assume $\overset{k}{w_1}, - \sim \overset{k}{w_1} \bullet, \overset{k}{w_2}, - \sim \overset{k}{w_2} \bullet$ for $-ne$ Trs. This is equivalent to $\overset{k}{-}, ne \sim ne; m \bullet$, which gives $\overset{k}{s}$ via the $\overset{k}{\wedge}$ -rules.

$$\frac{
 \begin{array}{c}
 \text{D} \quad \hat{\sim}_{s_1}^k \sim ne; m \bullet \\
 \text{D}_1 \\
 \vdots \\
 \text{D}_n
 \end{array}
 }{
 neE^{\hat{\dagger}} \bullet
 }$$

$\hat{\dagger} \bullet$ The occurrence at index m is not within the scope of \sim or n ; $k > N$; $\sim_{s_1}; \dots; \sim_{s_n} \bullet$.

Let $\hat{\sim}_s^k$ be a Hintikka formula. By classical completeness, there is such that for each k th Hintikka formula $\hat{\sim}_s^k$, there is a world $w > s$ such that $\hat{\sim}_w^k \cup \emptyset$.

Then it can be shown that $\emptyset \hat{\sim}_s^k, ne \sim ne; m \bullet$. Let $\hat{\sim}_s^k, ne \sim ne; m \bullet$.

Consider the case in which $s = 2$. Let $\hat{\sim}_{w_1}^k - \hat{\sim}_{w_2}^k$.

Assume $\hat{\sim}_{w_1}^k, - \sim_{w_1}^k \bullet \hat{\sim}_{w_2}^k, - \sim_{w_2}^k \bullet$ for $-ne$ Trs. This is equivalent to $\hat{\sim}_{w_1}^k, ne \sim ne; m \bullet$, which gives $\hat{\sim}_{w_1}^k$ via the $\hat{\sim}$ -rules.

Assume $\hat{\sim}_{w_1}^k, - \sim_{w_1}^k \bullet \hat{\sim}_{w_2}^k, ne \sim_{w_2}^k \bullet$ for $-ne$ Trs. This is equivalent to $\hat{\sim}_{w_2}^k, ne \sim ne; m \bullet$, which gives $\hat{\sim}_{w_2}^k$ by assumption.

$$\begin{array}{c}
 \text{D} \quad \wedge_{s_1}^k \sim ne; m \bullet \\
 \text{D}_1 \\
 \vdots \\
 \text{D}_n \quad \wedge_{s_n}^k \sim ne; m \bullet \\
 \hline
 neE^{\ddagger} \bullet
 \end{array}$$

$\wedge^{\ddagger} \bullet$ The occurrence at index m is not within the scope of \sim or n ; $k > N$; $\sim_{s_1}; \dots; \sim_{s_n} \bullet$.

Let \wedge_s^k be a Hintikka formula. By classical completeness, there is a world $w > s$ such that $\wedge_w^k \bullet$. Then it can be shown that $\wedge_s^k \bullet$.

Consider the case in which $\wedge_s^k \bullet$. Let $\wedge_{w_1}^k \bullet$ and $\wedge_{w_2}^k \bullet$ for $\sim ne$ Trs. This is equivalent to $\wedge_{w_1}^k \bullet, \sim \wedge_{w_2}^k \bullet$, which gives $\wedge_{w_1}^k \bullet$ via the \wedge -rules.

Assume $\wedge_{w_1}^k \bullet, \sim \wedge_{w_1}^k \bullet, \wedge_{w_2}^k \bullet, ne \sim \wedge_{w_2}^k \bullet$ for $\sim ne$ Trs. This is equivalent to $\wedge_{w_2}^k \bullet, ne \sim \wedge_{w_2}^k \bullet$, which gives $\wedge_{w_2}^k \bullet$ by assumption.

Similarly $\wedge_{w_1}^k \bullet, ne \sim \wedge_{w_1}^k \bullet, \wedge_{w_2}^k \bullet, \sim \wedge_{w_2}^k \bullet \bullet$ and $\wedge_{w_1}^k \bullet, ne \sim \wedge_{w_1}^k \bullet, \wedge_{w_2}^k \bullet, ne \sim \wedge_{w_2}^k \bullet \bullet$.

So $\wedge_s^k \bullet$ by iterated applications of $\sim ne$ Trs.

Completeness

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation f for a formula ϕ , each atom a is replaced by some \hat{a} such that $(\phi \rightarrow \hat{\phi})$:

$$\begin{array}{l}
 \hat{\phi} \\
 \phi \rightarrow \hat{\phi} \\
 \hat{p} \wedge \hat{q}, \hat{r} \vee \hat{s}
 \end{array}
 \qquad
 \begin{array}{l}
 f \\
 \hat{p} \wedge \hat{q}, \hat{r} \vee \hat{s}
 \end{array}$$

Completeness

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation f for a formula ϕ , each atom p is replaced by some \hat{p} such that $(\hat{p} \rightarrow p) \in \phi$:

$$\begin{array}{l}
 \hat{p} \rightarrow p, \hat{q} \rightarrow q \\
 \hat{p} \rightarrow p, \hat{q} \rightarrow q
 \end{array}
 \quad
 \begin{array}{l}
 \hat{p} \\
 \hat{q}
 \end{array}
 \quad
 \begin{array}{l}
 f \\
 \hat{p} \rightarrow p, \hat{q} \rightarrow q
 \end{array}$$

F : the set of all instantiations of ϕ

Completeness

The idea: simulate the disjunctive normal forms using "instantiations" [7]. In an instantiation f for a formula ϕ , each atom a is replaced by some \hat{a}_f such that $(\phi \rightarrow \bigwedge a \hat{a}_f) \in F$:

$$p \rightarrow q, \neg e \in F \implies \hat{p}_f \rightarrow \hat{q}_f, \neg e \in F$$

F : the set of all instantiations of ϕ

Since for each atom a we have $\hat{a}_f \in P$, where $P = \{ \hat{a}_f \mid a \in M; s \in SM; s \in \bullet \}$, then assuming that \vdash distributes over everything:

F

And given rules that simulate \vdash :

$$\text{if } \vdash \phi \rightarrow \psi \text{ then } \vdash \hat{\phi}_f \rightarrow \hat{\psi}_f$$

Problem: inBSML, 1 does not distribute over \cdot . For instance $n \wedge p \wedge 1 \wedge q \neq (n \wedge p) \wedge (1 \wedge q)$.

Problem: inBSML, \Box does not distribute over \vee . For instance $\Box(p \vee q) \not\equiv \Box p \vee \Box q$.

Solution: we treat maximal modal subformulas as atoms. $\Box(p \vee q) \equiv \Box p \vee \Box q$.

Problem: in BSML, \Box does not distribute over \vee . For instance $\Box(p \vee q) \not\equiv \Box p \vee \Box q$.

Solution: we treat maximal modal subformulas as atoms. $\Box(p \vee q) \equiv \Box p \vee \Box q$.

Lemma (1-distributive form): $\Box \phi \equiv \Box \psi$ implies $\phi \equiv \psi$ where ψ does not contain \Box within the scope of an \Box , and is in negation normal form.

Problem: in BSML, \Box does not distribute over \vee . For instance $\Box(p \vee q) \not\equiv \Box p \vee \Box q$.

Solution: we treat maximal modal subformulas as atoms. $\Box(p \vee q) \equiv \Box p \vee \Box q$.

Lemma (1-distributive form): $\Box \phi$ implies $\Box \psi$ where ψ does not contain \Box within the scope of an \Box , and is in negation normal form.

An instantiation σ of $\Box \psi$ in 1-distributive form:

each \Box is replaced by some \Box_{σ} where σ (ne

each $\Box \sim p$; $\Box p$; $\Box \neg p$; $\Box \neg \neg p$ is replaced by some \Box_{σ}^k where σ_{σ}^k (; k md[^] •)

Problem: in BSML, \Box does not distribute over \vee . For instance $\Box(p \vee q) \not\equiv \Box p \vee \Box q$.

Solution: we treat maximal modal subformulas as atoms. $\Box(p \vee q) \equiv \Box p \vee \Box q$.

Lemma (1-distributive form): $\Box \phi$ in BSML implies $\Box \psi$ where ψ does not contain \Box within the scope of an \Box , and is in negation normal form.

An instantiation σ of $\Box \psi$ in 1-distributive form:

each p is replaced by some ϕ where ϕ is

each $\Box p$; $\Box p$; $\Box p$; $\Box p$ is replaced by some ϕ where ϕ is

F

if $\Box \phi$; $\Box \phi$, then $\Box \phi$

\hat{O} (F (F

\hat{O} (F (F

$\vdash_f \vdash_k \text{Cmd}^{\wedge} \bullet \S sf_1; sf_2$
 $f \cup \emptyset \quad \begin{matrix} k \\ sf_1 \end{matrix} - \begin{matrix} k \\ sf_2 \end{matrix}$

\hat{O} (F (F

\hat{O} $\hat{sf}_1; sf_2 \bullet A$ $\overset{k}{sf_1} - \overset{k}{sf_2}$ ($\hat{sg}_1; sg_2 \bullet B$ $\overset{k}{sg_1} - \overset{k}{sg_2}$

$$\begin{array}{l}
 \vdash_f \vdash_k \text{Cmd}^{\wedge} \bullet \Ssf_1; sf_2 \\
 f \cup \emptyset \overset{k}{sf_1} - \overset{k}{sf_2}
 \end{array}$$

$$\hat{O} \quad (F (F$$

$$\hat{O} \quad \hat{sf}_1;sf_2 \bullet A \quad k_{sf_1} - k_{sf_2} (\quad \hat{sg}_1;sg_2 \bullet B \quad k_{sg_1} - k_{sg_2}$$

$$| f | k Cmd^ \bullet \Ssf_1;sf_2$$

$$f \cup \emptyset \quad k_{sf_1} - k_{sf_2}$$

$$k_{s_1} - k_{s_2} \quad k_{s_1>t}$$

$$tbs_2$$

$$| tbs_2 \quad k_{s_1>t} \emptyset \quad k_{s_1} - k_{s_2}$$

$$| tbs_2 \quad ; \quad k_{s_1>t} \emptyset$$

$$\hat{O} \quad ; \quad k_{s_1} - k_{s_2} \emptyset$$

$$\hat{O} \quad (F (F$$

$$\hat{O} \quad \hat{sf}_1;sf_2 \bullet A \quad \begin{matrix} k \\ sf_1 - \end{matrix} \begin{matrix} k \\ sf_2 \end{matrix} (\quad \hat{sg}_1;sg_2 \bullet B \quad \begin{matrix} k \\ sg_1 - \end{matrix} \begin{matrix} k \\ sg_2 \end{matrix}$$

$$\hat{O} \quad \hat{sf}_1;sf_2 \bullet A \quad tbsf_2 \quad \begin{matrix} k \\ sf_1 > t \end{matrix} (\quad \hat{sg}_1;sg_2 \bullet B \quad ubsg_2 \quad \begin{matrix} k \\ sg_1 > u \end{matrix}$$

$$\begin{matrix} | & f & | & k & Cmd^{\wedge} & \bullet & \Ssf_1;sf_2 \\ f & \cup \emptyset & \begin{matrix} k \\ sf_1 - \end{matrix} & \begin{matrix} k \\ sf_2 \end{matrix} \end{matrix}$$

$$\begin{matrix} \begin{matrix} k \\ s_1 - \end{matrix} & \begin{matrix} k \\ s_2 \end{matrix} & \begin{matrix} k \\ s_1 > t \end{matrix} \\ & tbs_2 & \\ | & tbs_2 & \begin{matrix} k \\ s_1 > t \end{matrix} \emptyset & \begin{matrix} k \\ s_1 - \end{matrix} & \begin{matrix} k \\ s_2 \end{matrix} \\ | & tbs_2 & ; & \begin{matrix} k \\ s_1 > t \end{matrix} \emptyset \\ & \hat{O} & ; & \begin{matrix} k \\ s_1 - \end{matrix} & \begin{matrix} k \\ s_2 \end{matrix} \emptyset \end{matrix}$$

$$\hat{O} \quad (F (F$$

$$\hat{O} \quad \begin{matrix} k_{sf_1} - k_{sf_2} \\ \hat{sf_1;sf_2} \bullet A \end{matrix} (\begin{matrix} k_{sg_1} - k_{sg_2} \\ \hat{sg_1;sg_2} \bullet B \end{matrix}$$

$$\begin{matrix} | f | k Cmd^ \bullet \Ssf_1;sf_2 \\ f \cup \emptyset \begin{matrix} k_{sf_1} - k_{sf_2} \end{matrix} \end{matrix}$$

$$\hat{O} \quad \begin{matrix} k_{sf_1>t} \\ \hat{sf_1;sf_2} \bullet A t b sf_2 \end{matrix} (\begin{matrix} k_{sg_1>u} \\ \hat{sg_1;sg_2} \bullet B u b sg_2 \end{matrix}$$

$$\hat{O} \quad | \hat{sf_1;sf_2} \bullet >A; t b sf_2 \quad \S \hat{sg_1;sg_2} \bullet >B; u b sg_2 \\ \begin{matrix} k_{sf_1>t} - k_{sf_1>t} \\ k_{sg_1>u} - k_{sg_1>u} \end{matrix}$$

$$\begin{matrix} k_{s_1} - k_{s_2} & k_{s_1>t} \\ t b s_2 & \emptyset \\ | t b s_2 & k_{s_1>t} \emptyset k_{s_1} - k_{s_2} \\ | t b s_2 & ; k_{s_1>t} \emptyset \\ \hat{O} & ; k_{s_1} - k_{s_2} \emptyset \end{matrix}$$

\hat{O} (F (F

\hat{O} $\hat{sf}_1; sf_2 \bullet > A$ $\begin{matrix} k \\ sf_1 - \end{matrix}$ $\begin{matrix} k \\ sf_2 \end{matrix}$ ($\hat{sg}_1; sg_2 \bullet > B$ $\begin{matrix} k \\ sg_1 - \end{matrix}$ $\begin{matrix} k \\ sg_2 \end{matrix}$

$\{ f \mid k \text{ Cmd}^{\wedge} \bullet \} \text{ ; } \{ sf_1; sf_2 \}$
 $f \cup \emptyset \begin{matrix} k \\ sf_1 - \end{matrix} \begin{matrix} k \\ sf_2 \end{matrix}$

\hat{O} $\hat{sf}_1; sf_2 \bullet > A$ t b $\begin{matrix} k \\ sf_1 > t \end{matrix}$ ($\hat{sg}_1; sg_2 \bullet > B$ u b $\begin{matrix} k \\ sg_1 > u \end{matrix}$

\hat{O} $\{ \hat{sf}_1; sf_2 \bullet > A; t \text{ b } \begin{matrix} k \\ sf_2 \end{matrix} \} \text{ ; } \{ \hat{sg}_1; sg_2 \bullet > B; u \text{ b } \begin{matrix} k \\ sg_2 \end{matrix} \}$
 $\begin{matrix} k \\ sf_1 > t \end{matrix} - \begin{matrix} k \\ sf_1 > t \end{matrix} \cup \emptyset \begin{matrix} k \\ sg_1 > u \end{matrix} - \begin{matrix} k \\ sg_1 > u \end{matrix} \cup \emptyset \begin{matrix} k \\ sg_1 - \end{matrix} \begin{matrix} k \\ sg_2 \end{matrix} \cup \emptyset \text{ g } \emptyset$

$\begin{matrix} k \\ s_1 - \end{matrix} \begin{matrix} k \\ s_2 \end{matrix}$ t b $\begin{matrix} k \\ s_2 \end{matrix}$ $\begin{matrix} k \\ s_1 > t \end{matrix}$
 $\{ t \text{ b } \begin{matrix} k \\ s_2 \end{matrix} \} \text{ ; } \{ \begin{matrix} k \\ s_1 > t \end{matrix} \} \cup \emptyset \begin{matrix} k \\ s_1 - \end{matrix} \begin{matrix} k \\ s_2 \end{matrix}$
 $\{ t \text{ b } \begin{matrix} k \\ s_2 \end{matrix} \} \text{ ; } \begin{matrix} k \\ s_1 > t \end{matrix} \cup \emptyset$
 $\hat{O} \text{ ; } \begin{matrix} k \\ s_1 - \end{matrix} \begin{matrix} k \\ s_2 \end{matrix} \cup \emptyset$

$$\hat{O} \quad (F (F$$

$$\hat{O} \quad \hat{sf}_1; sf_2 \bullet > A \quad \begin{matrix} k \\ sf_1 \end{matrix} - \begin{matrix} k \\ sf_2 \end{matrix} (\quad \begin{matrix} k \\ sg_1 \end{matrix} - \begin{matrix} k \\ sg_2 \end{matrix} \quad \hat{sg}_1; sg_2 \bullet > B$$

$$\begin{matrix} | & f & | & k & Cmd^{\wedge} & \bullet & \S sf_1; sf_2 \\ f & \dot{U} \emptyset & \begin{matrix} k \\ sf_1 \end{matrix} & - & \begin{matrix} k \\ sf_2 \end{matrix} & & \end{matrix}$$

$$\hat{O} \quad \hat{sf}_1; sf_2 \bullet > A \quad t b sf_2 \quad \begin{matrix} k \\ sf_1 > t \end{matrix} (\quad \hat{sg}_1; sg_2 \bullet > B \quad u b sg_2 \quad \begin{matrix} k \\ sg_1 > u \end{matrix}$$

$$\hat{O} \quad | \hat{sf}_1; sf_2 \bullet > A; t b sf_2 \quad \S \hat{sg}_1; sg_2 \bullet > B; u b sg_2$$

$$\begin{matrix} k \\ sf_1 > t \end{matrix} - \begin{matrix} k \\ sf_1 > t \end{matrix} \quad \begin{matrix} k \\ sg_1 > u \end{matrix} \quad \begin{matrix} k \\ sg_1 > u \end{matrix} \quad \emptyset \quad \begin{matrix} k \\ sg_1 \end{matrix} - \begin{matrix} k \\ sg_2 \end{matrix} \quad \emptyset \quad g \quad \emptyset$$

$$\begin{matrix} \begin{matrix} k \\ s_1 \end{matrix} - \begin{matrix} k \\ s_2 \end{matrix} & & \begin{matrix} k \\ s_1 > t \end{matrix} \\ & & t b s_2 \\ | t b s_2 & \begin{matrix} k \\ s_1 > t \end{matrix} \emptyset & \begin{matrix} k \\ s_1 \end{matrix} - \begin{matrix} k \\ s_2 \end{matrix} \\ | t b s_2 & ; \begin{matrix} k \\ s_1 > t \end{matrix} \emptyset & \\ \hat{O} & ; \begin{matrix} k \\ s_1 \end{matrix} - \begin{matrix} k \\ s_2 \end{matrix} \emptyset & \end{matrix}$$

$$\hat{O} \quad | \hat{sf}_1; sf_2 \bullet > A \quad \begin{matrix} k \\ sf_1 \end{matrix} - \begin{matrix} k \\ sf_2 \end{matrix} \emptyset$$

$$\hat{O} \quad (F (F$$

$$\hat{O} \quad \overset{k}{sf_1} - \overset{k}{sf_2} (\overset{k}{sg_1} - \overset{k}{sg_2}$$

$$! f ! k Cmd^ \bullet \S sf_1; sf_2$$

$$f \acute{U} \emptyset \overset{k}{sf_1} - \overset{k}{sf_2}$$

$$\hat{O} \quad \overset{k}{sf_1 > t} (\overset{k}{sg_1 > u}$$

$$\overset{k}{s_1} - \overset{k}{s_2} \quad \overset{k}{s_1 > t}$$

$$! t b s_2 \quad \overset{k}{s_1 > t} \emptyset \quad \overset{k}{s_1} - \overset{k}{s_2}$$

$$! t b s_2 \quad ; \overset{k}{s_1 > t} \emptyset$$

$$\hat{O} \quad ; \overset{k}{s_1} - \overset{k}{s_2} \emptyset$$

$$\hat{O} \quad ! \overset{k}{sf_1 > t} - \overset{k}{sf_1 > t} \acute{U} \emptyset \quad \overset{k}{sg_1 > u} \emptyset \quad \overset{k}{sg_1} - \overset{k}{sg_2} \emptyset \quad g \emptyset$$

$$\hat{O} \quad ! \overset{k}{sf_1} - \overset{k}{sf_2} \emptyset$$

$$\hat{O} \quad ! f > F \quad f \emptyset$$

$$\hat{O} \quad (F (F$$

$$\hat{O} \quad \hat{sf}_1;sf_2 \bullet > A \quad \begin{matrix} k \\ sf_1 \end{matrix} - \begin{matrix} k \\ sf_2 \end{matrix} (\quad \begin{matrix} k \\ sg_1 \end{matrix} - \begin{matrix} k \\ sg_2 \end{matrix} \hat{sg}_1;sg_2 \bullet > B$$

$$\begin{matrix} | & f & | & k & Cmd^{\wedge} & \bullet & \S sf_1;sf_2 \\ f & \dot{U} \emptyset & & \begin{matrix} k \\ sf_1 \end{matrix} & - & \begin{matrix} k \\ sf_2 \end{matrix} & \end{matrix}$$

$$\hat{O} \quad \hat{sf}_1;sf_2 \bullet > A \quad t b sf_2 \quad \begin{matrix} k \\ sf_1 > t \end{matrix} (\quad \hat{sg}_1;sg_2 \bullet > B \quad u b sg_2 \quad \begin{matrix} k \\ sg_1 > u \end{matrix}$$

$$\hat{O} \quad | \hat{sf}_1;sf_2 \bullet > A; t b sf_2 \quad \S \hat{sg}_1;sg_2 \bullet > B; u b sg_2$$

$$\begin{matrix} k \\ sf_1 > t \end{matrix} - \begin{matrix} k \\ sf_1 > t \end{matrix} \quad \begin{matrix} k \\ sg_1 > u \end{matrix} - \begin{matrix} k \\ sg_1 > u \end{matrix}$$

$$\begin{matrix} k \\ sf_1 > t \end{matrix} \dot{U} \emptyset \quad \begin{matrix} k \\ sg_1 > u \end{matrix} \emptyset \quad \begin{matrix} k \\ sg_1 \end{matrix} - \begin{matrix} k \\ sg_2 \end{matrix} \emptyset \quad g \emptyset$$

$$\begin{matrix} k \\ s_1 \end{matrix} - \begin{matrix} k \\ s_2 \end{matrix} \quad \begin{matrix} k \\ s_1 > t \end{matrix}$$

$$t b s_2$$

$$| t b s_2 \quad \begin{matrix} k \\ s_1 > t \end{matrix} \emptyset \quad \begin{matrix} k \\ s_1 \end{matrix} - \begin{matrix} k \\ s_2 \end{matrix}$$

$$| t b s_2 \quad ; \quad \begin{matrix} k \\ s_1 > t \end{matrix} \emptyset$$

$$\hat{O} \quad ; \quad \begin{matrix} k \\ s_1 \end{matrix} - \begin{matrix} k \\ s_2 \end{matrix} \emptyset$$

$$\hat{O} \quad | \hat{sf}_1;sf_2 \bullet > A \quad \begin{matrix} k \\ sf_1 \end{matrix} - \begin{matrix} k \\ sf_2 \end{matrix} \emptyset$$

$$\hat{O} \quad | \quad f > F \quad f \emptyset$$

$$\hat{O} \quad \emptyset$$

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